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THE COMPUTATION OF FERMI-DIRAC FUNCTIONS

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1—Introduction

The quantitative application of Fermi-Dirac statistics involves the evaluation of certain integrals which have not previously been tabulated. In this paper, tables are given of the values of the basic integrals most frequently required, with a view to placing Fermi-Dirac statistics on as firm a numerical basis as is Maxwell-Boltzmann statistics.

The expression for the energy distribution of particles subject to Fermi-Dirac statistics may be written in the form

$$\frac{dN}{d\epsilon} = \frac{\nu(\epsilon)}{e^{\alpha+\beta\epsilon}+1}, \tag{1·1}$$

where $\nu(\epsilon)$ is the number of states per unit energy range, and dN is the number of particles in the energy range ϵ to $\epsilon+d\epsilon$. In the statistical treatment, the parameters α and β, which are usually introduced as undetermined multipliers in a variational equation, are to be determined from two equations expressing conditions imposed by the total number of particles, and the total energy of the system. By linking up the statistical and thermodynamical treatments, interpretation can be given to α and β; this is expressed by

$$\beta = 1/kT, \quad \alpha = -\zeta/kT, \tag{1·2}$$

where ζ is the Gibbs free energy, or the chemical potential per particle. It is convenient to write

$$\eta = -\alpha, \tag{1·3}$$

when the distribution formula (1·1) becomes

$$\frac{dN}{d\epsilon} = \frac{\nu(\epsilon)}{e^{\epsilon/kT-\eta}+1}. \tag{1·4}$$

The advantages of the change of sign, expressed by (1·3), were pointed out when this symbolism was introduced (Stoner 1935); a distinctive symbol is convenient, and η seems suitable for the representation of a "reduced" energy in the sense of (1·2). In this paper we are concerned less with the immediate physical significance of η, however, than with its use as a convenient parameter specifying a particular distribution.

An energy distribution of states of particular importance is that in which

$$\nu(\epsilon) = C\epsilon^{\frac{1}{2}}, \tag{1·5}$$

where C is a constant, and the zero for ϵ is taken as that for the lowest energy state for a particle. This is the characteristic distribution when the energy is purely kinetic, and in particular it may be taken as applying, ordinarily with negligible error, to free electrons. The study of the distribution specified by (1·5), however, has a much wider range of useful application, for this distribution often holds very closely for electrons in partially filled energy bands in metals. When (1·5) holds, an implicit equation for η is

$$N = \int_0^\infty \frac{C\epsilon^{\frac{1}{2}}\,d\epsilon}{e^{\epsilon/kT-\eta}+1}. \tag{1·6}$$

In the state of lowest energy of the system, approached more closely the lower the temperature, the lowest energy states for the particles are completely occupied. It follows from (1·5) that the maximum particle energy ϵ_0 is then given by

$$N = \tfrac{2}{3}C\epsilon_0^{\frac{3}{2}}, \tag{1·7}$$

and (1·6) may therefore be written

$$N = \frac{3}{2}\cdot\frac{N}{\epsilon_0^{\frac{3}{2}}}\int_0^\infty \frac{\epsilon^{\frac{1}{2}}\,d\epsilon}{e^{\epsilon/kT-\eta}+1}, \tag{1·8}$$

while (1·4) becomes

$$\frac{dN}{d\epsilon} = \frac{3}{2}\cdot\frac{N}{\epsilon_0^{\frac{3}{2}}}\cdot\frac{\epsilon^{\frac{1}{2}}}{e^{\epsilon/kT-\eta}+1}. \tag{1·9}$$

The maximum particle energy, ϵ_0, is expressible, for free electrons, in terms of h, m and the concentration. It may not be possible to calculate ϵ_0 theoretically for more complicated systems, but generally, when (1·5) holds, it is convenient to use ϵ_0 rather than C to specify the system owing to its more immediate physical significance. The form of the integral in (1·8) is simplified by writing

$$x = \epsilon/kT, \tag{1·10}$$

when the equation becomes

$$N = \tfrac{3}{2}N\left(\frac{kT}{\epsilon_0}\right)^{\frac{3}{2}}\int_0^\infty \frac{x^{\frac{1}{2}}dx}{e^{x-\eta}+1}. \tag{1·11}$$

The evaluation of the integral in (1·11) for a particular value of η enables the corresponding value of (kT/ϵ_0) to be determined; from a series of such evaluations, the value of η corresponding to a particular value of (kT/ϵ_0) within the range considered may be found by inverse interpolation. If ϵ_0, a constant of the system, is known, the energy distribution of the particles at the particular temperature is then completely specified by (1·9). Although the precise determination of the energy distribution is not, in itself, of particular importance, it illustrates the application of the integrals very directly, and it is perhaps desirable to indicate the relation between the Fermi-Dirac calculation and the corresponding calculation using Maxwell-Boltzmann statistics.

Maxwell-Boltzmann statistics appears as a limit of Fermi-Dirac statistics,* as applied to the standard distribution of states specified by (1·5), for $-\eta \gg 1$, corresponding to $\epsilon_0/kT \ll 1$. The expression (1·11) then becomes

$$N = \tfrac{3}{2}N\left(\frac{kT}{\epsilon_0}\right)^{\frac{3}{2}}\int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{e^{x-\eta}},$$

$$= \tfrac{3}{2}N\left(\frac{kT}{\epsilon_0}\right)^{\frac{3}{2}}\cdot\frac{\sqrt{\pi}}{2}e^{\eta}, \qquad (1·12)$$

giving

$$e^{\eta} = \frac{4}{3\sqrt{\pi}}\left(\frac{\epsilon_0}{kT}\right)^{\frac{3}{2}}. \qquad (1·13)$$

By substitution, the usual Maxwellian distribution expression is obtained:

$$\frac{dN}{d\epsilon} = 2\pi N(\pi kT)^{-\frac{3}{2}}\epsilon^{\frac{1}{2}}e^{-\epsilon/kT}. \qquad (1·14)$$

In Maxwell-Boltzmann statistics, the integral in (1·12) is a gamma function, and a single integration gives an explicit expression for the parameter η in the distribution function in a form (1·13) which is applicable over the whole temperature range. In contrast, in Fermi-Dirac statistics, an extensive series of integrations is required to cover adequately the range from $\eta \gg 1$ to $-\eta \gg 1$, corresponding to the range from $(kT/\epsilon_0) \to 0$ to $(kT/\epsilon_0) \to \infty$.

The quantitative application of Fermi-Dirac statistics to systems of particles with the "standard" energy distribution of states (1·5) involves the evaluation of integrals which have the form

$$F_k(\eta) = \int_0^\infty \frac{x^k\,dx}{e^{x-\eta}+1}, \qquad (1·15)$$

especially for the values $k = \frac{1}{2}, \frac{3}{2}$. In particular, the distribution function, as shown above, leads to the integral

$$F_{\frac{1}{2}}(\eta) = \int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{e^{x-\eta}+1} = F(\eta) = F. \qquad (1·16)$$

The range of application of the functions (1·15) may be illustrated by reference to the thermal and magnetic properties of collective electrons in the theory of metals. When (1·5) is satisfied, the energy of the electrons may be expressed by

$$NkT\{F_{\frac{3}{2}}(\eta)/F_{\frac{1}{2}}(\eta)\} \quad \text{(Nordheim 1934)},$$

and the electronic specific heat may readily be calculated for the whole range of

* To avoid irrelevant complications, the symbol ϵ_0 has been retained in (1·12) and (1·13), but it should strictly be regarded as having merely the formal significance indicated by the equation (1·7); its interpretation as the maximum particle energy at absolute zero is applicable in Maxwell-Boltzmann statistics only when this is treated explicitly as a limit of Fermi-Dirac statistics.

temperature once the functions have been evaluated. The magnetic moment due to electron spin is given by

$$M = N(\mu^2 H/kT)\,(F'/F) \quad \text{(Stoner 1935)},$$

involving the derivative of F, if the first power only in H is considered; the inclusion of non-linear terms in H introduces terms in the higher order derivatives F'', F'''. Generally, the values of the functions (1·15) are required if numerical results are to be obtained from a theoretical investigation of the temperature variation of any collective electron property. Another field of application is in astrophysics, particularly in connexion with stars of the white dwarf type; for the interior of these stars consists effectively of a free electron gas at very high pressure.

The primary purpose of this paper is the evaluation of the function $F = F_{\frac{1}{2}}(\eta)$ for a wide range of values of the argument. From the table of $F_{\frac{1}{2}}(\eta)$ values, the $F_{\frac{3}{2}}(\eta)$ table is obtained by integration, as explained in § 6, while evaluation of the derivatives F', F'' etc. involves numerical differentiation. For negative values of η (corresponding roughly to $kT/\epsilon_0 > 1$) a rapidly convergent series for $F(\eta)$ may be obtained provided that $|\eta|$ is not too small. Further, for large positive values of η (i.e. $\eta \gg 1$, corresponding to $kT/\epsilon_0 \ll 1$), the integral may be represented by an asymptotic series. For intermediate values of η, however, no generally applicable series representation has been obtained. In treatments of collective electron susceptibility (Stoner 1935, 1936*a*) and specific heat (Stoner 1936*b*), use was made of the $F_k(\eta)$ series, or series derived from them, for high and low temperatures, and approximate results for the intermediate temperature range were obtained by graphical interpolation. Although for the purpose in view the rough values so obtained were perhaps adequate, the procedure was not very satisfactory, the range over which interpolation was necessary being considerable, and the degree of precision of the interpolated values somewhat uncertain. In any investigation in which something better than rough values for the intermediate region are required, particularly if differences of $F(\eta)$ are involved, the graphical interpolation method is quite inadequate. It was this consideration which led us to evaluate a number of integrals for the intermediate region by numerical quadrature. It is clearly unnecessary, so far as physical applications are concerned, to draw up an elaborate table of values of a function if these values are given with close approximation by a simple analytical expression. We found, however, that only for large values of $\eta(\eta \geq 16)$ could the asymptotic series give values comparable in precision with those we had obtained by numerical integration. There is no corresponding limitation to the precision obtainable from the series for $\eta \leq 0$; but here, for the smaller values of $|\eta|$ it is necessary to use a very large number of terms in the series, so that the incidental calculation of values which may be required is rather troublesome.

For these reasons, and in view of the basic importance of these integrals for physical applications, a systematic evaluation of $F(\eta)$ has been made for $-4{\cdot}0 \leq \eta \leq +20{\cdot}0$, so covering the range ordinarily required in applications. Values lying outside this range can be obtained with a precision comparable with that in the tables by using only two or three terms of the appropriate series. The computation of $F(\eta)$ values has been made by means of series at the outer parts of the range covered, and by numerical integration for intermediate values of η. The use of the series involves the evaluation of a number of coefficients which may have other applications; and for the asymptotic series, a numerical investigation of the degree of precision attainable has been made. In connexion with the numerical quadrature for a particular value of η, series expressions have been developed for the "head" (that is, the initial part of the x range) and "tail"; and for the larger values of η a modified procedure, which greatly reduces the labour involved in the numerical integration, has been adopted. As the most convenient procedure for evaluating the integral varies with the value of the argument, it is convenient to subdivide the account of the methods into sections dealing with different ranges of η values.

From the basic set of calculated values of $F(\eta)$, intermediate values have been obtained by interpolation, giving finally a table of $F(\eta)$ at intervals of 0·1 in the argument. From this table, successive derivatives are readily obtained, and these may be used for any further interpolation (direct or inverse) which may be required. The $F_{\frac{3}{2}}(\eta)$ values have been found by integration of $F(\eta)$, checks being provided by a number of direct evaluations of the function.

In the course of these computations we have made use of Barlow's tables* for the powers and the Smithsonian tables (Becker and van Orstrand 1909) for exponentials, the Smithsonian tables being supplemented when necessary by the extensive exponential tables of Newman (1883) and Glaisher (1883). The numerical work has been carried out with the aid of Brunsviga calculating machines.

2—Evaluation of $F(\eta)$ for $\eta<0$ and $\eta=0$

The function to be evaluated, namely

$$F_{\frac{1}{2}}(\eta)=\int_0^\infty \frac{x^{\frac{1}{2}}dx}{e^{x-\eta}+1}=F(\eta)=F, \tag{2·1}$$

is a member of the sequence†

$$F_k(\eta)=\int_0^\infty \frac{x^k dx}{e^{x-\eta}+1}. \tag{2·2}$$

Now it may readily be shown that, when $\eta\leqslant 0$,

$$F_k(\eta)=\Gamma(k+1)\sum_{r=1}^{\infty}(-)^{r-1}\frac{e^{r\eta}}{r^{k+1}}, \tag{2·3}$$

* "Barlow's Tables", ed. L. J. Comrie (London: Spon 1935). Sometimes more significant figures were required than are given in this edition. The extra figures were then calculated or obtained from the earlier edition.

† In the sequence with which we are concerned, in which k is half an odd integer, $F_k(\eta)$ is satisfactorily defined by the integral expression for k values down to $-\frac{1}{2}$. The appropriate generalization of the specification of $F_k(\eta)$ which will apply to a wider range of values of k is considered in an Appendix.

the series representation being appropriate for $k>-1$. In particular, for $k=\frac{1}{2}$,

$$F_{\frac{1}{2}}(\eta)=F(\eta)=F=\frac{\sqrt{\pi}}{2}\sum_{r=1}^{\infty}(-)^{r-1}\frac{e^{r\eta}}{r^{\frac{3}{2}}}. \tag{2·4}$$

This series for $F(\eta)$ has been evaluated for values of η from $-4{\cdot}0$ to $-0{\cdot}2$ at intervals of $0{\cdot}2$ in η, seven places of decimals being used in the calculations, and a number of terms summed sufficient to ensure accuracy to the sixth decimal place in the result.

An estimate of the maximum error in $F(\eta)$ when only n terms of the series are summed may be made by writing

$$F(\eta)=\frac{\sqrt{\pi}}{2}\sum_{r=1}^{n}(-)^{r-1}\frac{e^{r\eta}}{r^{\frac{3}{2}}}+R_n, \tag{2·5}$$

an upper limit to the remainder, R_n, being given by

$$R_n<\epsilon_a=\frac{\sqrt{\pi}}{2}\cdot\frac{e^{(n+1)\eta}}{(n+1)^{\frac{3}{2}}}. \tag{2·6}$$

The values of ϵ_a, the maximum absolute error, and of ϵ_r, the maximum relative error (i.e. $\epsilon_a/F(\eta)$), as dependent on the number, n, of terms used, are shown below for $\eta=-4{\cdot}0$. ($F(-4{\cdot}0)=0{\cdot}016\,127\,74$.)

n	ϵ_a	ϵ_r
1	0·000 105 11	$6{\cdot}5\times10^{-3}$
2	0·000 001 05	$6{\cdot}5\times10^{-5}$
3	0·000 000 01	$7{\cdot}7\times10^{-7}$
4	0·000 000 00	$1{\cdot}1\times10^{-8}$

Thus, beyond the negative limit ($\eta=-4{\cdot}0$) to the η range covered by the table, an absolute accuracy corresponding to that in the tabulated $F(\eta)$ values is obtained by using only 2 terms in the series (2·4), the relative error being less than 10^{-2}, 10^{-4} and 10^{-6} for 1, 2 and 3 terms. The number of terms, say n', required to give the $F(\eta)$ values to a specified accuracy increases as $|\eta|$ decreases, changing very rapidly for η values between $-1{\cdot}0$ and $-0{\cdot}2$. This is illustrated by the following series of values of n' corresponding to an accuracy of 1 in the seventh decimal place:

η	$-4{\cdot}0$	$-3{\cdot}0$	$-2{\cdot}0$	$-1{\cdot}0$	$-0{\cdot}8$	$-0{\cdot}6$	$-0{\cdot}4$	$-0{\cdot}2$
n'	3	4	7	13	16	21	30	56

Partly as a check on the integration method, the $F(\eta)$ values for $-1{\cdot}0\leqslant\eta\leqslant-0{\cdot}2$ have also been determined by numerical integration, satisfactory agreement being obtained.* Even in the extreme example with $\eta=-0{\cdot}2$, however, the series method is considerably less laborious than the numerical integration method.

* A further method of evaluating $F(\eta)$, appropriate for $|\eta|\leqslant 0{\cdot}3$, is indicated in § 7.

$\eta = 0$. In the evaluation of $F(\eta)$ when $\eta = 0$, the direct summation of a number of terms of the expansion (2·2) is not convenient, but $F_{\frac{1}{2}}(0)$ may be expressed as a multiple of a Riemann zeta function,* for

$$F_{\frac{1}{2}}(0) = \tfrac{1}{2}\sqrt{\pi}\sum_{r=1}^{\infty}(-)^{r-1}r^{-\frac{3}{2}},$$

$$= \tfrac{1}{2}\sqrt{\pi}(1-2^{-\frac{1}{2}})\sum_{r=1}^{\infty}r^{-\frac{3}{2}}, \tag{2·7}$$

$$= \tfrac{1}{2}\sqrt{\pi}(1-2^{-\frac{1}{2}})\,\zeta(\tfrac{3}{2}). \tag{2·8}$$

As the values of the zeta function are given to only four significant figures in the Jahnke-Emde tables (1933), $\zeta(\frac{3}{2})$ was evaluated from the positive series in (2·7) by making use of the Euler-Maclaurin formula (Whittaker and Robinson 1932, p. 165)

$$\frac{1}{w}\int_a^{a+rw} f(x)\,dx = (f_0+f_1+\dots+f_r) - \tfrac{1}{2}(f_0+f_r) - \frac{w}{12}(f_r'-f_0') + \frac{w^3}{720}(f_r'''-f_0''') - \frac{w^5}{30240}(f_r^{(5)}-f_0^{(5)})\dots, \tag{2·9}$$

in which f_r is written for $f(a+rw)$. By taking the upper limit of the integral in this formula to be large, we obtain

$$\zeta(\tfrac{3}{2}) = \sum_{r=1}^{a} r^{-\frac{3}{2}} + 2a^{-\frac{1}{2}} + a^{-\frac{3}{2}}\left\{\frac{1}{2}+\frac{1}{8a}-\frac{7}{384}\cdot\frac{1}{a^3}+\frac{11}{1024}\cdot\frac{1}{a^5}\dots\right\}. \tag{2·10}$$

Checks are provided by carrying out the summation for different values of the integer a. Subsequently we obtained and made use of the ten decimal place tables of the zeta function calculated by Gram (1925).

The $F(\eta)$ values, obtained to seven places of decimals at intervals of 0·2 in η, have been checked by the method of differences, and the interval in η has been reduced to 0·1 by interpolating values calculated from the Bessel formula,†

$$f_n = f_0 + n\delta_{\frac{1}{2}} + B_2(\delta_0^2+\delta_1^2) + B_3\delta_{\frac{1}{2}}^3 + B_4(\delta_0^4+\delta_1^4)\dots, \tag{2·11}$$

using the Bessel coefficients‡ for $n = \frac{1}{2}$:

$$B_2 = -\tfrac{1}{16}, \quad B_3 = 0, \quad B_4 = \tfrac{3}{256}. \tag{2·12}$$

While we cannot claim accuracy for the seventh place digit, it probably has some significance, and in tabulating the values, rounded to six places, an indication can be given of the next digit. The method adopted throughout this paper is the use of a dot

* The relations between the values of $F_k(0)$ and the Riemann functions $\zeta(k+1)$ are discussed in § 7.

† The notation adopted in the central difference formulae in this paper is similar to that used by Comrie (1936) except that a small δ is used in place of the italicized capital Δ, and an ordinary number for the index in place of dashes or Roman numbers.

‡ Tables of Bessel coefficients are given by Comrie (1936).

which indicates that the digit following the last one printed lies between 3 and 7; e.g. 0·019 670˙ is a number lying between 0·019 670 3 and 0·019 670 7. This is more convenient than the use of such an alternative form as 0·019 670½. The listed values of $F(\eta)$ for $-4{\cdot}0 \leqslant \eta \leqslant 0{\cdot}0$ are believed to be correct to within 0˙.

3—Evaluation of $F(\eta)$ for $0{\cdot}0 < \eta < 3{\cdot}0$

For the range $0{\cdot}0 < \eta < 3{\cdot}0$, we have found no general* method of evaluating $F(\eta)$ other than by direct numerical integration, supplemented by the use of series for the contributions from the initial and final parts of the x range (the "head" and "tail"). The range of integration was suitably subdivided, and the value of the quotient (evaluated to six or seven decimal places) of $x^{\frac{1}{2}}$ by $(e^{x-\eta}+1)$ was obtained at appropriate equal intervals for each part of the range. The intervals in the several parts of the x range were so chosen that the contributions to the integral from the fourth order differences were small, and quite negligible from differences of higher order. As illustrative, the following ranges and intervals (w) in x were found suitable for $\eta = 1{\cdot}0$: $x < 0{\cdot}40$ ($w = 0{\cdot}02$), 0·40–2·00 (0·05), 2·0–4·0 (0·1), 4·0–9·0 (0·2), 9·0–17·5 (0·5); the quotients being determined to six decimal places up to $x = 9{\cdot}0$, and to seven places for the range 9·0–17·5. The central difference formula (Whittaker and Robinson 1932, p. 147)†

$$\frac{1}{w}\int_a^{a+rw} f(x)\,dx = (f_0+f_1+\ldots+f_r)-\tfrac{1}{2}(f_0+f_r)-\tfrac{1}{12}(\delta_r-\delta_0)+\tfrac{11}{720}(\delta_r^3-\delta_0^3)\ldots, \qquad (3{\cdot}1)$$

where $\delta_r = \frac{1}{2}(\delta_{r-\frac{1}{2}}+\delta_{r+\frac{1}{2}})$, and f_r is written for $f(a+rw)$, was employed in carrying out the integration. As described below, series summation methods were developed for evaluating the contributions to the integral sum from the initial part (0–0·08) of the x range, where (3·1) is inapplicable, and from the final part (9·0–∞), where the series method is less troublesome.

3*a*—*Series summation method for head*

The formula (3·1) is inapplicable to the initial part of the x range, since the higher order differences for $r = 0$ are not available. Further, owing to the occurrence of $x^{\frac{1}{2}}$ in the integrand, the use of a formula involving forward differences is not convenient unless very small intervals are used, since the terms in the formula converge very slowly. These difficulties have been overcome by developing series formulae for the quadrature in the region of $x = 0$.

* A method described in § 7 is applicable for small values of η, but the values of certain Riemann zeta functions are required.

† See footnote † on p. 73.

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The contribution to $F(\eta)$ from the range $x=0$ to $x=\alpha(\alpha<1)$ may be expressed in terms of $\lambda=e^{\eta}$:

$$\int_0^{\alpha}\frac{x^{\frac{1}{2}}\,dx}{e^{x-\eta}+1}=\frac{2}{3}\cdot\frac{\lambda}{\lambda+1}\alpha^{\frac{3}{2}}\{1-a_1\alpha-a_2\alpha^2-a_3\alpha^3-a_4\alpha^4\ldots\},\tag{3·2}$$

where $$a_1=\frac{3}{5}\cdot\frac{1}{1+\lambda},\quad a_2=\frac{3}{14}\cdot\frac{\lambda-1}{(1+\lambda)^2},\quad a_3=\frac{1}{18}\cdot\frac{\lambda^2-4\lambda+1}{(1+\lambda)^3},$$

$$a_4=\frac{1}{88}\cdot\frac{(\lambda-1)(\lambda^2-10\lambda+1)}{(1+\lambda)^4}.$$

This series was used to give the contribution from the range $x=0$ to $x=0\cdot08$, or, for the higher values of η, to $x=0\cdot10$. A check was obtained by using a larger value for $\alpha(0\cdot20)$ and comparing the difference between the two results with that found by numerical integration.

3*b*—*Series summation method for tail*

For the smaller values of η, the contribution to $F(\eta)$ from the tail can be estimated with adequate precision without extending unduly the range of integration, as illustrated by the details given above for $\eta=1$. In general, however, and particularly for the larger values of η, it is convenient and more satisfactory to evaluate the tail contribution by a series method. A suitable series may be obtained in terms of incomplete gamma functions:

$$\int_{\beta}^{\infty}\frac{x^{\frac{1}{2}}\,dx}{e^{x-\eta}+1}=\int_{\beta}^{\infty}\frac{x^{\frac{1}{2}}e^{\eta-x}\,dx}{1+e^{\eta-x}}$$

$$=\sum_{r=1}^{\infty}(-)^{r-1}\int_{\beta}^{\infty}x^{\frac{1}{2}}e^{r(\eta-x)}\,dx.\tag{3·3}$$

For large values of β, the asymptotic series representation of the integrals in (3·3), obtainable by successive integration by parts, is appropriate:

$$\int_{\beta}^{\infty}x^{\frac{1}{2}}e^{-rx}\,dx=\frac{1}{r}\beta^{\frac{1}{2}}e^{-r\beta}B_r(\beta),\tag{3·4}$$

where $$B_r(\beta)=1+\frac{1}{2r\beta}\left\{1+\sum_{s=1}^{\infty}(-)^s\frac{1.3.5\ldots(2s-1)}{(2r\beta)^s}\right\},\tag{3·5}$$

so that $$\int_{\beta}^{\infty}\frac{x^{\frac{1}{2}}\,dx}{e^{x-\eta}+1}=\beta^{\frac{1}{2}}\sum_{r=1}^{\infty}(-)^{r-1}\frac{1}{r}B_r(\beta)e^{-r(\beta-\eta)}.\tag{3·6}$$

In using the series (3·6) for $0\cdot2\leqslant\eta\leqslant3\cdot0$ with $\beta=9$, not more than three terms were

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required, and the evaluation of the appropriate coefficients $B_r(9)$ from (3·5) involved not more than ten terms of the series. The values of these coefficients are given below:

$$B_1(9) = 1{\cdot}052\,878, \quad B_2(9) = 1{\cdot}027\,063, \quad B_3(9) = 1{\cdot}018\,193. \tag{3·7}$$

The uncertainties arising from the use of an asymptotic series representation were ± 5 in the sixth decimal place in B_1, and less than $0^{\cdot}$ in B_2 and B_3; the error to which these might give rise in the contribution to $F(\eta)$ was negligible (<1 in the seventh decimal place for $\eta = 3$).

The function $F(\eta)$ has been determined at intervals of 0·2 in η in the range $0{\cdot}0 < \eta < 3{\cdot}0$ by the methods described in this section. The interval in the argument was subsequently reduced to 0·1 by interpolation, using the Bessel formula (2·11). It is believed that the $F(\eta)$ values in this range, which are tabulated to six decimal places, are certainly correct to 1 in the sixth decimal place and may actually be better than this degree of precision suggests, so that the indication of the next digit given by the dot symbol $^{\cdot}$ is not without significance.

4—Evaluation of $F(\eta)$ for $\eta \geqslant 3$

The evaluation of $F(\eta)$ for the larger values of η by straightforward numerical integration with an accuracy comparable with that obtained in the lower range would have been extremely laborious. The determination of this function for a single value of η involves drawing up a table of entries of $x^{\frac{1}{2}}$ and $(e^{x-\eta}+1)$, the machine work in making the divisions, differencing to the third order to check the quotients, the integration summation, and further checking. Even when a reasonably systematic procedure has been developed, so that full advantage is taken of the possibility of using common entries for a series of calculations, and replacing divisions by multiplications, the evaluation of $F(\eta)$, even for a small value of η (say $\eta = 2$, with $F(\eta) \doteqdot 2{\cdot}5$), may require about eight hours. As η increases, so do the values of the integrand and the range in which it is appreciable, and the volume of work increases considerably; consequently a modified method for the computation of $F(\eta)$ has been employed, the method being particularly useful for large values of η, though no advantage is gained by adopting it for values of η less than about 3.

This method consists in the calculation of the difference between $F(\eta)$ and its approximate value, $\frac{2}{3}\eta^{\frac{3}{2}}$, and reduces very materially, certainly by more than a half, the time required for the evaluation of the integral. The appropriate transformation of the integral follows.

$$F(\eta) = \int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{e^{x-\eta}+1} = \int_0^\eta x^{\frac{1}{2}}\,dx + \int_0^\eta x^{\frac{1}{2}}\left\{\frac{1}{e^{x-\eta}+1}-1\right\}dx + \int_\eta^\infty \frac{x^{\frac{1}{2}}\,dx}{e^{x-\eta}+1},$$

$$= \tfrac{2}{3}\eta^{\frac{3}{2}} + \int_\eta^{2\eta} \frac{x^{\frac{1}{2}}\,dx}{e^{x-\eta}+1} - \int_0^\eta \frac{x^{\frac{1}{2}}\,dx}{1+e^{\eta-x}} + \int_{2\eta}^\infty \frac{x^{\frac{1}{2}}\,dx}{e^{x-\eta}+1}. \tag{4·1}$$

By setting $y = x-\eta$ and $y = \eta-x$ in the first and second integrals respectively in (4·1), these become

$$\int_{\eta}^{2\eta} \frac{x^{\frac{1}{2}}\,dx}{e^{x-\eta}+1} = \int_0^{\eta} \frac{(\eta+y)^{\frac{1}{2}}\,dy}{e^y+1},$$

and

$$\int_0^{\eta} \frac{x^{\frac{1}{2}}\,dx}{e^{\eta-x}+1} = \int_0^{\eta} \frac{(\eta-y)^{\frac{1}{2}}\,dy}{e^y+1}.$$

Whence

$$F(\eta) = \tfrac{2}{3}\eta^{\frac{3}{2}} + \int_0^{\eta} \frac{\{(\eta+y)^{\frac{1}{2}}-(\eta-y)^{\frac{1}{2}}\}\,dy}{e^y+1} + \int_{2\eta}^{\infty} \frac{x^{\frac{1}{2}}\,dx}{e^{x-\eta}+1},$$

which may be written

$$F(\eta) = \tfrac{2}{3}\eta^{\frac{3}{2}} + \int_0^{\eta} \phi(y)\,dy + \int_{2\eta}^{\infty} f(x)\,dx. \tag{4·2}$$

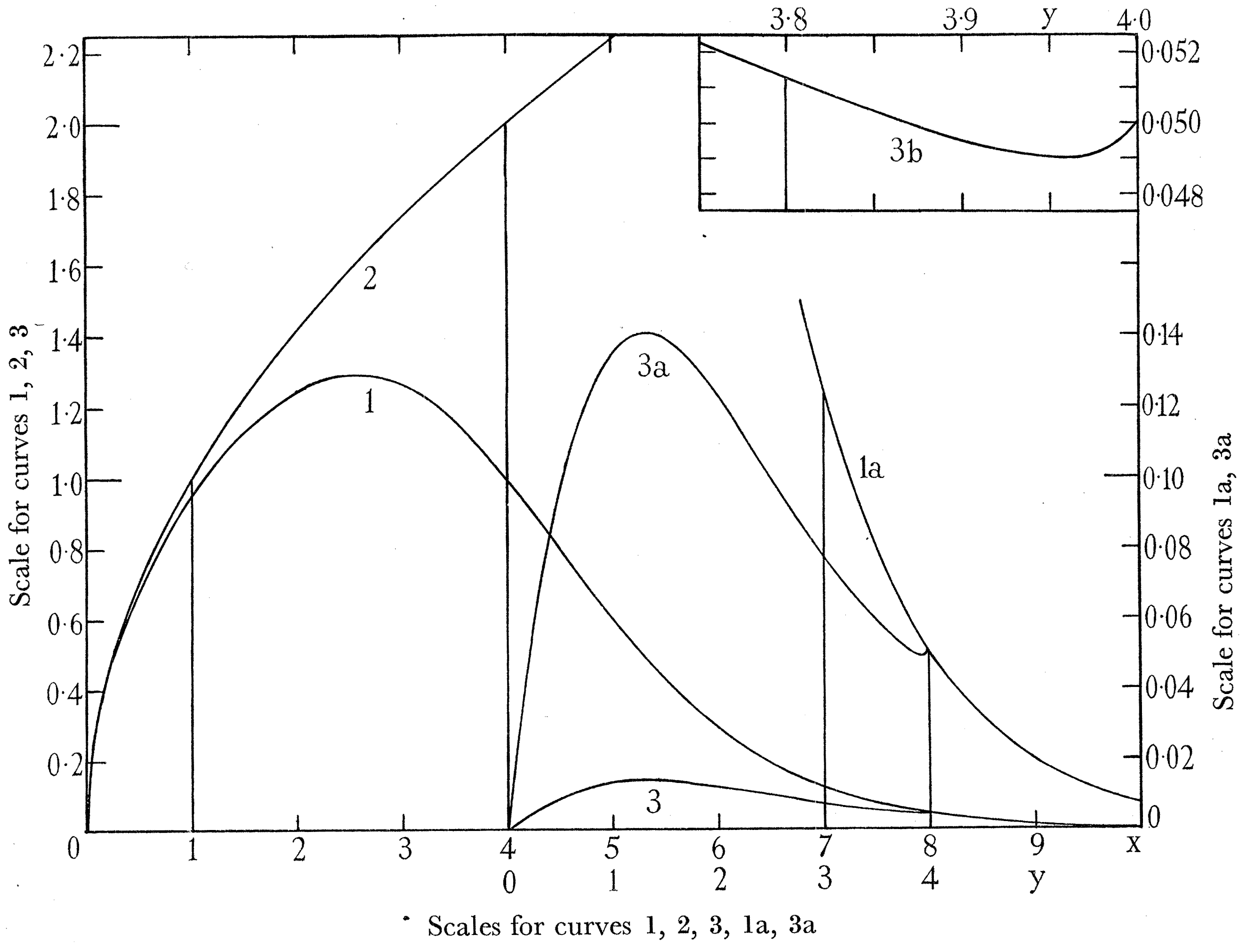

Fig. 1—Graphs of $f(x) = x^{\frac{1}{2}}/(e^{x-\eta}+1)$ and of $\phi(y) = \{(\eta+y)^{\frac{1}{2}}-(\eta-y)^{\frac{1}{2}}\}/(e^y+1)$ for $\eta = 4·0$, and of $x^{\frac{1}{2}}$. 1, $f(x)$; 1a, $f(x)$ with larger scale for ordinates; 2, $x^{\frac{1}{2}}$; 3, $\phi(y)$; 3a, $\phi(y)$ with larger scale for ordinates; 3b, $\phi(y)$ in region $y = \eta$, with larger scale for ordinates and abscissae.

The character and purpose of this transformation will be more readily apparent from an inspection of fig. 1, in which graphs of $f(x) = x^{\frac{1}{2}}/(e^{x-\eta}+1)$ for $\eta = 4·0$ (curve 1), and of $x^{\frac{1}{2}}$ (curve 2), are shown. The second integral in (4·2) is the area under the tail of the $f(x)$ curve from $x=2\eta$. The first integral corresponds to the difference of the area under the $f(x)$ curve from $x=\eta$ to $x=2\eta$ and that between the $x^{\frac{1}{2}}$ and $f(x)$ curves from $x=0$ to $x=\eta$; it is the area under the $\phi(y)$ curve (curve 3) between the limits $y=0$ and $y=\eta$.

To carry out the $\phi(y)$ quadratures, a series of square roots were written down on two separate strips in opposite order and the differences required for each value of η were obtained. The denominators, (e^y+1), required for any value of η, are common to the calculations for that and all higher values of η, so that considerable time could be saved by obtaining reciprocals. As the $\phi(y)$ entries were much smaller than the $f(x)$ entries in the $\eta<3{\cdot}0$ calculations, larger intervals in y were used (0·05, 0·1 or 0·2—an interval of 0·2 was used throughout the integration for $\eta\geqslant 8{\cdot}0$), the range of integration being subdivided into parts so chosen that contributions to the integral sum from fourth and higher order differences were small and negligible respectively. No difficulty arises in the application of the central difference formula (3·1) in the $y=0$ region, as the table of $\phi(y)$ entries can be extended to negative values of y; but in the $y=\eta$ region, the second and higher differences of $\phi(y)$ increase rapidly, and equation (3·1) cannot be applied without error. This is due to the fact that when $y=\eta$ and $x=2\eta$ the two functions $\phi(y)$ and $f(x)$ $(\phi(\eta)=f(2\eta))$ do not join smoothly; for $(\eta-y)^{\frac{1}{2}}$ behaves near $y=\eta$ like $x^{\frac{1}{2}}$ near $x=0$, and the $\phi(y)$ curve shows a "spur" in the neighbourhood of $y=\eta$. This spur is apparent in curve 3a of fig. 1, and is shown on a larger scale in curve 3b. (Actually the minimum of $\phi(y)$ becomes less marked, and occurs closer to $y=\eta$ the greater the value of η.) It becomes necessary to limit the $\phi(y)$ quadrature to the range $y=0$ to $y=\eta-\alpha$, choosing α so that the region in which the differences of $\phi(y)$ increase rapidly is excluded. The working formula

$$F(\eta)=\tfrac{2}{3}\eta^{\frac{3}{2}}+H(\alpha)+I(\alpha)+T(\alpha), \tag{4·3}$$

where

$$H(\alpha)=\int_0^\alpha x^{\frac{1}{2}}\left\{\frac{1}{e^{x-\eta}+1}-1\right\}dx,$$

$$I(\alpha)=\int_0^{\eta-\alpha}\phi(y)\,dy,$$

$$T(\alpha)=\int_{2\eta-\alpha}^{\infty}f(x)\,dx,$$

is obtained from (4·2) by replacing $\int_{\eta-\alpha}^{\eta}\phi(y)\,dy$ by the integral $H(\alpha)$ and $\int_{2\eta-\alpha}^{2\eta}f(x)\,dx$. (In fig. 1, the vertical lines at $x=1$ and $x=7\,(y=3)$ denote the limits of integration for $H(\alpha)$ and $I(\alpha)$ respectively when $\alpha=1$; the contributions were also calculated with $\alpha=0{\cdot}2$, and the corresponding limit is indicated in curve 3b.) The integrals $I(\alpha)$ were evaluated by means of the formula (3·1), and the $H(\alpha)$ and $T(\alpha)$ contributions were determined by the series summation methods described below.

4 *a*—*Series summation method for the contribution* $H(\alpha) = \int_0^\alpha x^{\frac{1}{2}}\left\{\frac{1}{e^{x-\eta}+1}-1\right\}dx$

Two formulae have been used for the evaluation of the H contribution. The first is appropriate to the adoption of a small value of α, e.g. $\alpha = 0{\cdot}2$, and is suitable for any value of η. The second is convenient for use with larger values of α, provided that $(\eta-\alpha)$ is not small, and with the choice $\alpha = 1$, may conveniently be applied for $\eta \geqslant 4{\cdot}0$.

(i) The integrand of $H(\alpha)$ differs from the integrand of equation (3·2) by $x^{\frac{1}{2}}$, so that

$$H(\alpha) = -\frac{2}{3}\cdot\frac{\lambda}{\lambda+1}\alpha^{\frac{3}{2}}\left\{\frac{1}{\lambda}+a_1\alpha+a_2\alpha^2+a_3\alpha^3+a_4\alpha^4\ldots\right\}, \tag{4·4}$$

where $\lambda = e^\eta$, and the coefficients are those given in (3·2).

(ii) Another series representation of $H(\alpha)$ with decreasing coefficients may be obtained by expanding the integrand of H in a series of powers of $\exp\{-(\eta-\alpha)\}$:

$$H(\alpha) = \tfrac{2}{3}\alpha^{\frac{3}{2}}\sum_{r=1}^{\infty}(-)^r A_r(\alpha)\,e^{-r(\eta-\alpha)}, \tag{4·5}$$

where

$$A_r(\alpha) = 3e^{-r\alpha}\sum_{s=0}^{\infty}\frac{1}{2s+3}\cdot\frac{(r\alpha)^s}{s!}. \tag{4·6}$$

The coefficients $A_r(\alpha)$ involve $\int_0^\alpha x^{\frac{1}{2}}e^{rx}\,dx$ and are obtainable in the series form (4·6) by expanding the exponential in the integrand and integrating term by term. With $\alpha = 1$, values of $A_r(\alpha)$, which are given below, may be determined correctly to the seventh decimal place by using 10, 13, 16 and 18 terms in the series for $r = 1, 2, 3$ and 4 respectively:

$$A_1(1) = 0{\cdot}692\,880\,6,\quad A_2(1) = 0{\cdot}510\,004\,3,$$
$$A_3(1) = 0{\cdot}394\,894\,1,\quad A_4(1) = 0{\cdot}318\,495\,3. \tag{4·7}$$

The series (4·5) converges rapidly for the larger values of η, and within the bounds of 4·0–5·2, 5·2–6·4 and 6·4–10·0 for η, only 4, 3 and 2 terms respectively are required; while for $\eta \geqslant 10{\cdot}0$, the term in A_1 is sufficient. As compared with (4·4) the series (4·6) has the advantage that the same coefficients $A_r(\alpha)$ apply for all values of η. It is particularly useful for large η values ($\eta \geqslant 7{\cdot}2$) when the lower limit of the integral $T(\alpha)$ in (4·3) is so large that the $T(\alpha)$ contribution may be determined without appreciable error by means of equation (3·6).

4 *b*—*Evaluation of the contribution* $T(\alpha) = \int_{2\eta-\alpha}^{\infty}\frac{x^{\frac{1}{2}}\,dx}{e^{x-\eta}+1}$

For $\eta < 7{\cdot}2$, the integral $T(\alpha)$ was evaluated by numerical integration; or, the range of integration was divided, and the first part of the T contribution found by numerical quadrature, while the further part was determined by the series method, using the

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formula (3·6) with appropriate values of the lower limit β. (The values used were $\beta = 11$ for the η range 5·2–6·0 and $\beta = 13$ for the range 6·2–7·0.) For $\eta \geqslant 7{\cdot}2$ the series summation alone was used, two terms only of the series (3·6) being required for $7{\cdot}2 \leqslant \eta \leqslant 9{\cdot}0$ and one term for $\eta > 9{\cdot}0$. Thus for $\eta \geqslant 7{\cdot}2$, the evaluation of $F(\eta)$ involves numerical integration only for $I(\alpha)$ in (4·3), $H(\alpha)$ and $T(\alpha)$ both being easily determined by series summations.

Table 4·1 illustrates the magnitudes of the contributions to $F(\eta)$ in that range of the argument to which the modified method of this section has been applied. It shows the value of the method in that the degree of precision attained in the value of $F(\eta)$ is determined effectively by the accuracy to which the I contribution is evaluated. For $\eta = 10{\cdot}0$, for example, the absolute accuracy of $F(\eta) = 21{\cdot}344\,471$ is effectively the same as that of $I(\alpha) = 0{\cdot}262\,125^{\cdot}$.

Table 4·1—Contributions to $F(\eta)$ for $\eta = 4$, 6, 8 and 10.

$$F(\eta) = \tfrac{2}{3}\eta^{\frac{3}{2}} + H(\alpha) + I(\alpha) + T(\alpha). \quad \alpha = 1$$

η	4	6	8	10
$\frac{2}{3}\eta^{\frac{3}{2}}$	5·333 333˙	9·797 959	15·084 945˙	21·081 851
H	−0·022 186	−0·003 097	−0·000 421	−0·000 057
I	0·322 294˙	0·326 177˙	0·292 318	0·262 125˙
T	0·137 285	0·023 245˙	0·003 644	0·000 552
$H+I+T$	0·437 393˙	0·346 326	0·295 541	0·262 620˙
$F(\eta)$	5·770 727	10·144 285	15·380 486˙	21·344 471

The value of $F(\eta)$ given here for $\eta = 4$ differs by 0˙ in the sixth place from the value in the final table. This is due to the rounding of the listed contributions to the sixth place. In the whole of this work, estimates of contributions to $F(\eta)$ were made to at least one decimal place more than was to be retained in the final listed value.

The methods described in this section have been applied to the determination of $F(\eta)$ at intervals of 0·2 in η in the range $3{\cdot}0 \leqslant \eta \leqslant 8{\cdot}0$, 0·4 in the range $8{\cdot}0 \leqslant \eta \leqslant 12{\cdot}0$, and then at integral values of η up to $\eta = 16$. This ensured an overlap with the range for which the asymptotic series method, discussed in § 5, can give results with a comparable degree of precision. Interpolations were first made, using the Bessel formula (2·11), to intervals of 0·2 for $8{\cdot}0 \leqslant \eta \leqslant 16{\cdot}0$, from the values of $F(\eta) - \frac{2}{3}\eta^{\frac{3}{2}}$; and then over the range $3{\cdot}0 \leqslant \eta \leqslant 16{\cdot}0$ to 0·1 intervals, from the $F(\eta)$ values. (The further differences required in interpolating in the neighbourhood of $\eta = 16{\cdot}0$ were obtained from the $F(\eta)$ values for $\eta > 16{\cdot}0$, calculated only by the asymptotic series method.) At each stage of the interpolation, the values were checked by the method of differences. For $3{\cdot}0 \leqslant \eta \leqslant 4{\cdot}0$, the $F(\eta)$ values are given to six places of decimals and are believed to be correct to within 1 in the sixth place. Only five places of decimals are listed for $4{\cdot}0 \leqslant \eta \leqslant 16{\cdot}0$; but as these are rounded values they should be correct to within 0˙ in the fifth place.

5—Asymptotic series expansion for $F(\eta)$

An asymptotic series expansion for the integral,

$$F_k(\eta) = \int_0^\infty \frac{x^k\,dx}{e^{x-\eta}+1}, \tag{5·1}$$

was first given by Sommerfeld (1928), and in a somewhat generalized treatment Nordheim (1934) considered the conditions for validity. Subject to an error of the order $\exp(-\eta)$, an expression appropriate when $\eta \gg 1$ is

$$F_k(\eta) = \frac{\eta^{k+1}}{k+1}\left\{1+\sum_{r=1}^{n} a_{2r}\eta^{-2r}\right\}+R_{2n}, \tag{5·2}$$

where
$$a_{2r} = 2c_{2r}(k+1)\,k\ldots(k-r+2), \tag{5·3}$$

and
$$c_{2r} = \sum_{s=1}^{\infty}(-)^{s-1}s^{-2r} = (1-2^{1-2r})\,\zeta(2r). \tag{5·4}$$

Gilham (1936) has shown that for the remainder term,

$$R_{2n} < L_{2n} = (2n+2)\,a_{2n+2}\,\eta^{k-2n-1}. \tag{5·5}$$

The evaluation of $F_k(\eta)$ as given by (5·2) has been discussed briefly in a previous paper (Stoner 1936*b*) with particular reference to the number of terms to be used to obtain the best approximation, and the values of the coefficients c_{2r}, for $r \leqslant 3$, have been given to eight significant figures (Stoner 1936*a*). In the previous work, the accuracy aimed at could be attained by using only two or three terms of the series, but in the present work it is necessary to use more terms to obtain the required degree of precision (an accuracy of 1 unit in the sixth decimal place in the value of $F(\eta)$) even for large values of η. The necessary coefficients and the corresponding zeta function values (taken from Gram's table) are given in Table 5·1.

Table 5·1—Zeta functions and coefficients in asymptotic series expansion for $F(\eta)$

$2r$	$\zeta(2r)$	c_{2r}	a_{2r}
2	1·644 934 07	0·822 467 03	1·233 700 5
4	1·082 323 23	0·947 032 83	1·065 411 9
6	1·017 343 06	0·985 551 09	9·701 518 5
8	1·004 077 36	0·996 232 88	242·715 02
10	1·000 994 58	0·999 039 51	11 865·691
12	1·000 246 09	0·999 757 69	958 843·43

In applying the expansion (5·2), the best approximation to $F_k(\eta)$ is obtained by choosing n such that the remainder term R_{2n} is a minimum; in particular, in evaluating the series for $F(\eta) = F_{\frac{1}{2}}(\eta)$, the precision is increased by using terms up to the nth (i.e. in taking n terms under the summation sign in (5·2)), provided that

$$(2n+2)\, a_{2n+2} < 2n\, a_{2n}\, \eta^2,$$

or

$$\eta > \left\{\frac{n+1}{n} \cdot \frac{a_{2n+2}}{a_{2n}}\right\}^{\frac{1}{2}} = \left\{\frac{n+1}{4n}\,(3-4n)\,(1-4n)\,\frac{c_{2n+2}}{c_{2n}}\right\}^{\frac{1}{2}}. \tag{5·6}$$

The values of n which will give the best approximation to $F(\eta)$ for successive intervals in the range of η values are set out below:

η	1·314		3·696		5·776		7·817		9·847		11·87
$2n$		2		4		6		8		10	
η	11·87		13·89		15·90		17·91		19·92		21·93
$2n$		12		14		16		18		20	

Thus for values of η lying between 11·87 and 13·89, the best approximation is obtained by using six terms of the series. Values of the upper limit to the error, ϵ_a, in $F(\eta)$, due to summing over only n terms in (5·2), are given by L_{2n} in (5·5) with $k = \frac{1}{2}$, and are set out in Table 5·2 together with the corresponding values of the limit to the relative error, ϵ_r.

Table 5·2 indicates the precision obtainable by means of the series for any part of the η range. For $\eta = 2, 4, 6, 8$, for example, the maximum precision is 20, 1·3, 0·11 and 0·011 %, and is attained by using 1, 2, 3 and 4 terms respectively in the summation. The increase in the error due to using more terms than the number indicated by the criterion (5·6) is illustrated by the ϵ values set down for these values of η corresponding to the use of one further term, namely 69, 2·7, 0·19 and 0·016 %. The precision becomes comparable with that obtained by the methods described in previous sections when $\eta \geq 16{\cdot}0$, and is then attained by using five terms in the summation. Although this is less than the optimum number of terms, as given by (5·6), the decrease in the error with further increase in number of terms up to the optimum is slight, and no useful purpose is served by calculating terms beyond the sixth, using five for the summation and the sixth in calculating the error. (It should be pointed out that the inherent error in (5·2), of order $\exp(-\eta)$, is smaller than that given by (5·5), and that in the range $\eta > 16{\cdot}0$, in which $F(\eta)$ has been calculated only by the asymptotic series method, it is completely negligible.)

The values of $F(\eta)$ have been calculated at intervals of 0·2 in η for the range $10{\cdot}0 \leq \eta \leq 20{\cdot}0$ from the asymptotic series, with $n = 5$. A comparison with the values calculated by the modified numerical integration method of § 4 showed that the actual errors in the values given by (5·2) are considerably less than the upper limits to the errors given in Table 5·2. For $\eta = 10, 12, 14, 16$, for example, the differences F(integration) $-F$(series) are approximately 8, 5, 1 and 0 in the sixth decimal place, as compared with the upper limits to the error in F(series) of 242, 36, 7 and 2 respectively. It has been assumed, therefore, that for $\eta \geqslant 16{\cdot}0$ the asymptotic series method gives $F(\eta)$ correctly to six decimal places, and the function has been evaluated by this method only for $\eta > 16{\cdot}0$ at intervals of 0·2. As described in § 4, the $F(\eta)$ values for the range $10{\cdot}0 \leqslant \eta \leqslant 16{\cdot}0$ were obtained at intervals in η of 0·2 by interpolation from the values calculated by the numerical integration method, checks being provided by comparison with the $F(\eta)$ values given by the series expansion calculations. The complete set of $F(\eta)$ values for $10{\cdot}0 \leqslant \eta \leqslant 20{\cdot}0$, at intervals of 0·2 in η, was checked by the method of differences; the intervals were then reduced to 0·1, and the final set of values again checked. As the basic $F(\eta)$ values obtained by the modified integration method for $10{\cdot}0 \leqslant \eta \leqslant 16{\cdot}0$ are certainly correct to 1 in the sixth decimal place, and the $F(\eta)$ values obtained by the series method for $\eta > 16{\cdot}0$ have at least this degree of precision, all the $F(\eta)$ values for $10{\cdot}0 \leqslant \eta \leqslant 20{\cdot}0$ may be accepted as correct to 1 or at most 2 in the sixth place. The tabulated values are given to five places of decimals, the dot symbol ˙ giving an indication of the next digit as explained in § 2.

Table 5·2—Upper limits to the error in the value of $F_{\frac{1}{2}}(\eta)$ given by equation (5·2) with $k=\frac{1}{2}$, made by using only n terms under the summation sign

Above: Absolute error, ϵ_a, in units of the sixth decimal place. Below: Relative error, ϵ_r, the number in brackets indicating the negative power of 10. Thus 3·2 (3) indicates a relative error of $3{\cdot}2 \times 10^{-3}$.

η \ $2n$	2	4	6	8	10
2	502 240	1 715 002			
	2·0 (1)	6·9 (1)			
4	88 784	75 793	158 018		
	1·5 (2)	1·3 (2)	2·7 (2)		
6	32 219	12 224	11 327	19 227	
	3·2 (3)	1·2 (3)	1·1 (3)	1·9 (3)	
8	15 695	3 349	1 746	1 667	2 525
	1·0 (3)	2·2 (4)	1·1 (4)	1·1 (4)	1·6 (4)
10	8 985	1 227	409	250	242
	4·2 (4)	5·8 (5)	1·9 (5)	1·2 (5)	1·1 (5)
12	5 696	540	125	53	36
	2·0 (4)	1·9 (5)	4·5 (6)	1·9 (6)	1·3 (6)
14	3 874	270	46	14	7
	1·1 (4)	7·7 (6)	1·3 (6)	4·1 (7)	2·0 (7)
16	2 775	149	19	5	2
	6·5 (5)	3·5 (6)	4·5 (7)	1·1 (7)	4·1 (8)
18	2 067	87	9	2	1
	4·0 (5)	1·7 (6)	1·8 (7)	3·3 (8)	1·0 (8)
20	1 589	54	5	1	0
	2·7 (5)	9·0 (7)	7·6 (8)	1·2 (8)	2·9 (9)

6—Derivatives and integrals of the function $F(\eta)$

Derivatives of $F(\eta)$—As pointed out in § 1, in applications there are a number of functions closely related to $F(\eta)$ which are frequently required, among them the derivatives F', F'', etc. From the values of $F(\eta)$, listed at an interval (w) of 0·1 in η, for $-4{\cdot}0 \leqslant \eta \leqslant 20{\cdot}0$, the first derivatives are readily evaluated by means of the formula*

$$wf_0' = \delta_0 - \tfrac{1}{6}\delta_0^3 + \tfrac{1}{30}\delta_0^5 \ldots, \tag{6·1}$$

or, in more convenient form,

$$wf_0' = \delta_{-\frac{1}{2}} + \tfrac{1}{2}\delta_0^2 - \tfrac{1}{12}(\delta^3_{-\frac{1}{2}} + \delta^3_{+\frac{1}{2}}) + \tfrac{1}{60}(\delta^5_{-\frac{1}{2}} + \delta^5_{+\frac{1}{2}}) \ldots. \tag{6·2}$$

The values of $wF'(\eta)$ have been obtained over the complete range of η values given in the table by using the first three terms of (6·2), the contribution of the δ^5 terms being negligible to six decimal places. The w^2F'' values have been found by applying (6·2) to the entries in the wF' table rather than by applying a formula for second order derivatives in terms of differences of the original $F(\eta)$ values. This procedure is convenient, as it is desirable to find the differences of the wF' entries for checking purposes, and it has the advantage that it smooths out, to some extent, the effect of any irregularities in the higher differences of $F(\eta)$ due to rounding errors in the entries. A similar method has been adopted in deriving the w^3F''' table. In carrying out the numerical work, one more figure than the number tabulated has been used throughout, and the tabulated wF', w^2F'' and w^3F''' values are believed to be correct to the same degree of precision as the corresponding $F(\eta)$ values, namely to within $0^{\cdot}$ and 1 in the sixth decimal place for the ranges $-4{\cdot}0 \leqslant \eta \leqslant 0{\cdot}0$ and $0{\cdot}0 < \eta \leqslant 4{\cdot}0$ respectively, and to within $0^{\cdot}$ in the fifth place for $4{\cdot}0 < \eta \leqslant 20{\cdot}0$.

The need for the derivatives of $F(\eta)$ in applications is a sufficient reason for their tabulation. It is, however, convenient to list the values of wF', w^2F'', w^3F''' rather than those of the derivatives themselves, since the method adopted gives $w^rF^{(r)}$ with the same precision as the original F values. If the values of the derivative are required, it is merely necessary to multiply the corresponding tabulated values $w^rF^{(r)}$ by 10^r, since $w = 0{\cdot}1$ throughout the table. (In physical applications, a much smaller absolute accuracy in successive derivatives than in F is usually adequate; that provided by the table will be more than sufficient for most purposes.) In addition to their immediate use in applications, the $w^rF^{(r)}$ values serve the purpose of a table of differences for the $F(\eta)$ entries, in that they may be used directly for interpolation by the methods described in § 8. The smoothing out of the effect of rounding errors in $F(\eta)$, mentioned above, here gives the set of $w^rF^{(r)}$ values, calculated as described, a definite advantage over a table of differences. Moreover, the equivalent of a table of differences is provided at once for $F'(\eta)$ and $F''(\eta)$ as well as for $F(\eta)$.

The Function $\frac{2}{3}F_{\frac{3}{2}}(\eta)$—The evaluation of $F_{\frac{3}{2}}(\eta)$ from $F(\eta)$ is complementary to the

* See footnote † on p. 73.

evaluation of the derivatives of $F(\eta)$, in that it involves integration of $F(\eta)$. The sequence of functions $F_k(\eta)$ is satisfactorily defined by (2·2) for $k > -1$; hence for $k > 0$

$$F'_k(\eta) = \frac{\partial}{\partial\eta}F_k(\eta) = \int_0^\infty x^k \frac{\partial}{\partial\eta}\left(\frac{1}{e^{x-\eta}+1}\right)dx = -\int_0^\infty x^k\frac{\partial}{\partial x}\left(\frac{1}{e^{x-\eta}+1}\right)dx$$

$$= \int_0^\infty \frac{kx^{k-1}\,dx}{e^{x-\eta}+1},$$

i.e.

$$F'_k(\eta) = kF_{k-1}(\eta).^* \tag{6·3}$$

In the sequence defined by half-odd-integral values of k, (6·3) applies for all $k \geq \frac{1}{2}$, and, in particular,

$$F'_{\frac{3}{2}}(\eta) = \tfrac{3}{2}F_{\frac{1}{2}}(\eta) = \tfrac{3}{2}F(\eta),$$

so that

$$\tfrac{2}{3}F_{\frac{3}{2}}(\eta) = \int_a^\eta F(\eta)\,d\eta + \tfrac{2}{3}F_{\frac{3}{2}}(a), \tag{6·4}$$

where a is any suitably chosen limit of integration. The method adopted to compute the $\frac{2}{3}F_{\frac{3}{2}}(\eta)$ values was to evaluate $\frac{2}{3}F_{\frac{3}{2}}(a)$ with $a = -4·0$, using the series (2·3), and to add the values of $\int F(\eta)\,d\eta$ as given by the central difference formula (3·1), or the Euler-Maclaurin formula (2·9), which is convenient when the values of the derivatives and their differences are known. In order to provide checks to this lengthy series of calculations, the values of $F_{\frac{3}{2}}(a)$ were determined directly for other values of a, namely, -3, -2, -1, 0, 5, 16, 20. The methods adopted were those used for evaluating $F(\eta)$ for similar values of the argument; the appropriate modifications of the formulae are briefly described below.

For $a < 0$, the series (2·3) was employed, while for $a = 0$, use was made of the relation (cf. 2·8)

$$F_{\frac{3}{2}}(0) = \tfrac{3}{4}\sqrt{\pi}(1-2^{-\frac{3}{2}})\,\zeta(\tfrac{5}{2}). \tag{6·5}$$

Complete agreement was obtained between the value of $\frac{2}{3}F_{\frac{3}{2}}(0)$ as calculated from (6·4) with $a = -4·0$, and from (6·5), to one decimal place beyond that given in the final table. To check the integration between $\eta = 0$ and $\eta = 5$, the function $F_{\frac{3}{2}}(a)$ was calculated for $a = 5$ by the modified method of § 4. We may write (cf. (4·2) and (4·3))

$$F_{\frac{3}{2}}(\eta) = \tfrac{2}{5}\eta^{\frac{5}{2}} + H'(\alpha) + I'(\alpha) + T'(\alpha), \tag{6·6}$$

where

$$H'(\alpha) = \int_0^\alpha x^{\frac{3}{2}}\left\{\frac{1}{e^{x-\eta}+1} - 1\right\}dx,$$

$$I'(\alpha) = \int_0^{\eta-\alpha}\frac{\{(\eta+y)^{\frac{3}{2}} - (\eta-y)^{\frac{3}{2}}\}\,dy}{e^y+1},$$

$$T'(\alpha) = \int_{2\eta-\alpha}^\infty \frac{x^{\frac{3}{2}}\,dx}{e^{x-\eta}+1},$$

* A more general derivation of this recurrence relation is given in an Appendix.

and develop series expansions for the $H'(\alpha)$ and $T'(\alpha)$ contributions as in § 4. Thus, corresponding to the expansion (4·5), we find

$$H'(\alpha) = \tfrac{2}{5}\alpha^{\frac{5}{2}} \sum_{r=1}^{\infty} (-)^r e^{-r(\eta-\alpha)}\{1-A_r(\alpha)\}/r, \tag{6·7}$$

which can be evaluated for $\eta = 5$ by taking $\alpha = 1$ and using the values for the coefficients $A_r(1)$ given by (4·7). By carrying out a calculation similar to that in § 3*b*, we find, as a formula suitable for the determination of the T' contribution (cf. (3·6)),

$$\int_{\beta}^{\infty} \frac{x^{\frac{3}{2}}\,dx}{e^{x-\eta}+1} = \beta^{\frac{1}{2}} \sum_{r=1}^{\infty} (-)^{r-1} \frac{1}{r}\left\{\beta + \frac{3}{2r} B_r(\beta)\right\} e^{-r(\beta-\eta)}, \tag{6·8}$$

which enables $T'(\alpha)$ to be evaluated by taking $\beta = 9$ and using the coefficients $B_r(9)$ of (3·7). In the computation of the $I'(\alpha)$ contribution, the central difference formula (3·1) was used. This calculation is much more troublesome than the corresponding one in determining $F(\eta)$ owing to the necessity of evaluating a series of $\frac{3}{2}$ powers and also because of the larger numbers involved. The contributions (rounded to six decimal places) to $F_{\frac{3}{2}}(5)$ given by these calculations are shown in Table 6·1; the resulting value of $\frac{2}{3}F_{\frac{3}{2}}(\eta)$, $18{\cdot}534\ 964_{06}$, is in good agreement with that obtained from (6·4) with $a = 0$, namely, $18{\cdot}534\ 964_{12}$.

Table 6·1—Contributions to $F_{\frac{3}{2}}(\eta)$ for $\eta = 5$.
$F_{\frac{3}{2}}(\eta) = \frac{2}{5}\eta^{\frac{5}{2}} + H'(\alpha) + I'(\alpha) + T'(\alpha)$. $\alpha = 1$

$\frac{2}{5}\eta^{\frac{5}{2}}$	22·360 680
H'	−0·005 544
I'	4·870 868˙
T'	0·576 442
$H'+I'+T'$	5·441 766˙
$F_{\frac{3}{2}}(\eta)$	27·802 446˙

The equation (5·2) gives an asymptotic series expansion suitable for the computation of $F_{\frac{3}{2}}(a)$ for $a = 16$ and 20, and the coefficients (5·3) for $k = \frac{3}{2}$ are simple rational fractions of the corresponding coefficients used in the $k = \frac{1}{2}$ calculation, and given in Table 5·1. By summing to five terms, the upper limit to the error in using the series expansion is

$$\epsilon'_a = \tfrac{2}{5}\eta^{\frac{5}{2}} \times 12 \times \tfrac{5}{19} a_{12}\eta^{-12}. \tag{6·9}$$

In the calculations, therefore, the absolute error is greater than in the corresponding $F(\eta)$ calculation (the ratio being $3\eta/19$), although the relative error is smaller (in the ratio $\frac{5}{19}$). The agreement obtained between the values of $\frac{2}{3}F_{\frac{3}{2}}(\eta)$ for $\eta = 16$ and 20, calculated from the asymptotic series expansion, and from equation (6·4), with $a = 5$, is shown by the following comparisons:

		Numerical integration	Asymptotic series
$\frac{2}{3}F_{\frac{3}{2}}(\eta)$ values:	$\eta = 16$	279·638 884	279·638 883˙ (±3)
	$\eta = 20$	484·378 857	484·378 856 (±0˙)

Although not exact to the sixth decimal place, the agreement is certainly as good as could be expected, and, indeed, provides a most satisfactory check on the calculations for a wide range of η values; in view of the number of figures to be given in the tables, the differences are of no importance.

In carrying out the calculations of $\frac{2}{3}F_{\frac{3}{2}}(\eta)$, at least one more figure was used than appears in the listed values, and the precision should be similar to that of the corresponding $F(\eta)$ values, namely 0· and 1 in the sixth place for $-4{\cdot}0 \leqslant \eta \leqslant 0{\cdot}0$, and $0{\cdot}0 < \eta \leqslant 4{\cdot}0$ respectively, and 0· in the fifth place for $4{\cdot}0 < \eta \leqslant 20{\cdot}0$. The values of $\frac{2}{3}F_{\frac{3}{2}}(\eta)$ are tabulated rather than those of $F_{\frac{3}{2}}(\eta)$, as F, F', F'', F''' are successive derivates of $\frac{2}{3}F_{\frac{3}{2}}(\eta)$, and the tabulated functions may be used directly in interpolation processes (see §8); convenience in applications is little affected, as other numerical factors usually occur in formulae involving $F_{\frac{3}{2}}(\eta)$.

7—Some characteristics of the tabulated functions

It is perhaps desirable to discuss certain general characteristics of the functions tabulated in this paper. The curves in fig. 2 indicate the variation with η of the functions $\frac{2}{3}F_{\frac{3}{2}}(\eta)$, $F_{\frac{1}{2}}(\eta) = F$, F', F'' and F''', for values of η lying between -3 and $+5$. The functions represented by the full curves may all be expressed as $(d/d\eta)^r F_{\frac{1}{2}}(\eta)$, and the curves are labelled according to the value of r. (The $\frac{2}{3}F_{\frac{3}{2}}(\eta)$ curve shown for $r=-1$ implies the choice of an appropriate integration constant.) The broken curve represents a limiting function to which all the other functions tend, as explained below, for large negative η.

Consideration of the behaviour of these functions for very large negative and positive values of η, and in the neighbourhood of $\eta = 0$, enables a survey to be made of the variation of the functions over all values of the argument.

Behaviour of functions as $\eta \to -\infty$—The limiting form of $F(\eta) = F_{\frac{1}{2}}(\eta)$ when $\eta \to -\infty$ is given by equation (2·4) as

$$[F(\eta)]_{\eta\to-\infty} \to \tfrac{1}{2}\pi^{\frac{1}{2}} e^{\eta}; \tag{7·1}$$

further, repeated application of the relation

$$F'_k(\eta) = kF_{k-1}(\eta), \tag{6·3}$$

which is valid for $k > 0$, shows that, for positive r,

$$F^{(r)}_{r+\frac{1}{2}}(\eta) = \{(r+\tfrac{1}{2})\,(r+\tfrac{1}{2}-1)\ldots\tfrac{3}{2}\}\,F(\eta). \tag{7·2}$$

In the limit $\eta \to -\infty$, therefore, the relation between the functions $F_k(\eta)$ is the same as that between the gamma functions $\Gamma(k+1)$, and the limiting forms of

$$\ldots, \tfrac{2}{5}\cdot\tfrac{2}{3}F_{\frac{5}{2}}, \tfrac{2}{3}F_{\frac{3}{2}}, F, F', F'', \ldots$$

are all $\frac{1}{2}\pi^{\frac{1}{2}}e^{\eta}$. The approach to equality with increasing value of $(-\eta)$ is indicated in fig. 2, and appears also in the tables. (At $\eta = -4{\cdot}0$, $\frac{1}{2}\pi^{\frac{1}{2}}e^{\eta} = 0{\cdot}016\,231\,7$.)

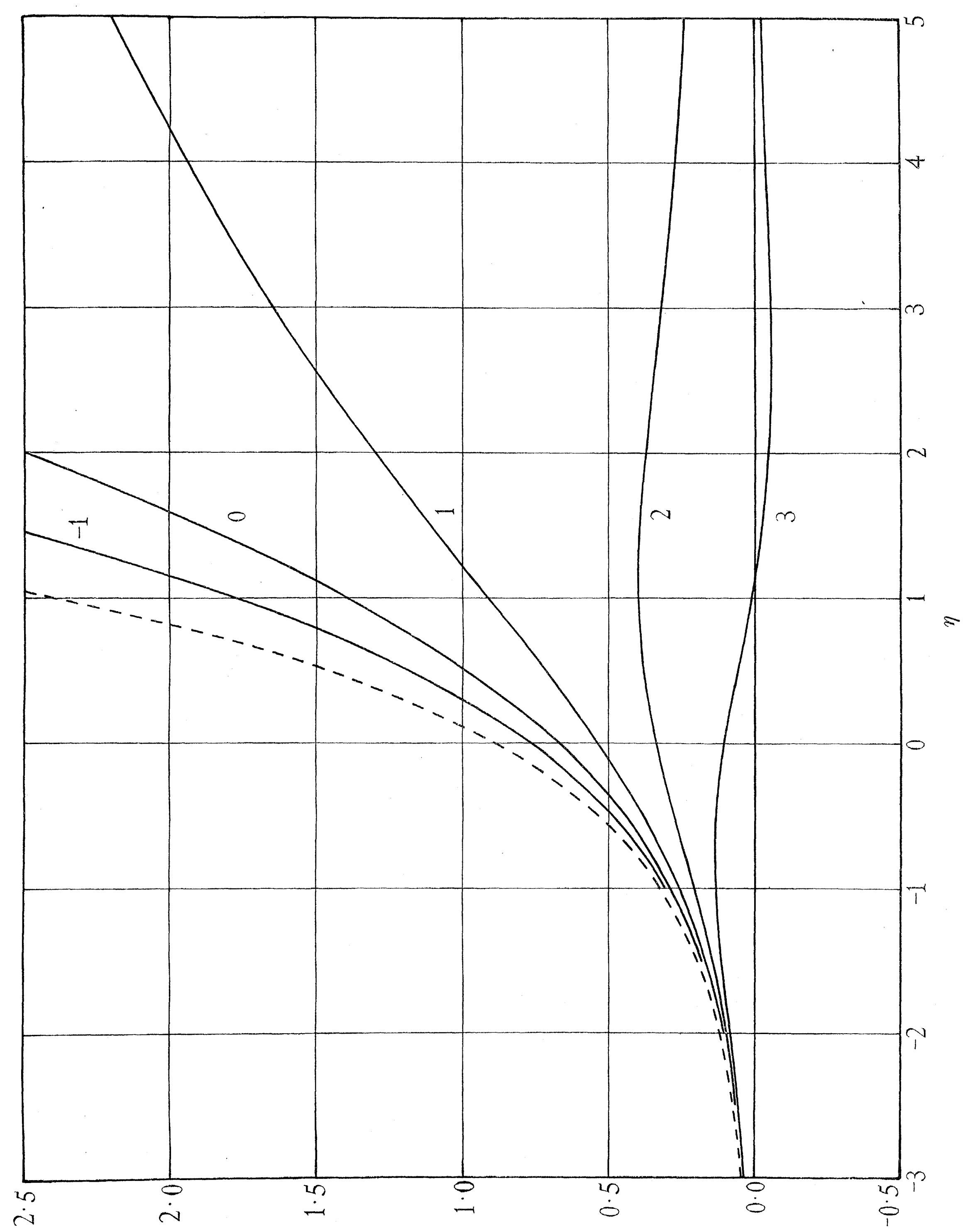

Fig. 2—$F_{\frac{1}{2}}(\eta)$ and related functions:

$$F_k(\eta) = \int_0^\infty \{x^k/(e^{x-\eta}+1)\}\,dx; \quad F_{\frac{1}{2}}(\eta) = F = \tfrac{2}{3}F'_{\frac{3}{2}}(\eta).$$

Full curves: (-1), $\tfrac{2}{3}F_{\frac{3}{2}}(\eta)$; (0), $F_{\frac{1}{2}}(\eta) = F$; (1), F'; (2), F''; (3), F'''.

Broken curve: $\tfrac{1}{2}\pi^{\frac{1}{2}}e^{\eta}$.

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Behaviour of functions as $\eta \to \infty$—The limiting form of $F_k(\eta)$ when $\eta \to \infty$ is given by the asymptotic series expansion (5·2) as

$$[F_k(\eta)]_{\eta\to\infty} \to \frac{1}{k+1}\eta^{k+1}, \tag{7·3}$$

so that the limiting forms of $\frac{2}{3}F_{\frac{3}{2}}(\eta)$, $F_{\frac{1}{2}}(\eta) = F$, F', F'' and F''' are $\frac{2}{3}\cdot\frac{2}{5}\eta^{\frac{5}{2}}$, $\frac{2}{3}\eta^{\frac{3}{2}}$, $\eta^{\frac{1}{2}}$, $\frac{1}{2}\eta^{-\frac{1}{2}}$ and $-\frac{1}{2}\cdot\frac{1}{2}\eta^{-\frac{3}{2}}$ respectively.

The general character of the functions changes from that of an exponential to that of a power as η changes from $-\infty$ to $+\infty$, and it is this change which is responsible for the difficulty in evaluating the functions in the transition region. The functions are all positive for large negative η values, and all increase initially as η increases; but whereas the functions F', F and $F_{\frac{3}{2}}$ (and all functions F_k with $k > \frac{1}{2}$ in the sequence of half-odd-integral k values) increase monotonically with increase in η, the higher order derivatives $F^{(r)}(\eta)$ with $r \geq 2$ exhibit $(r-2)$ zeros, and the values of the functions for large η are positive or negative according as $(r-2)$ is even or odd.

Behaviour of functions in the neighbourhood of $\eta = 0$—Reference has already been made to the connexion between the $F_k(0)$ values and the Riemann zeta function (see equations (2·8) and (6·5)). By means of the identity

$$\frac{1}{e^x+1} = \frac{1}{e^x-1} - \frac{2}{e^{2x}-1}, \tag{7·4}$$

the relation,

$$F_k(0) = \int_0^\infty \frac{x^k\,dx}{e^x+1} = (1-2^{-k})\int_0^\infty \frac{x^k\,dx}{e^x-1} = (1-2^{-k})\,\Gamma(k+1)\,\zeta(k+1), \tag{7·5}$$

is obtained as valid for $k > 0$ (corresponding to the range in which $\zeta(s)$ may be defined by $\sum_{n=1}^{\infty} n^{-s}$). Differentiation under the integral sign shows that

$$F_k'(0) = (1-2^{1-k})\,\Gamma(k+1)\,\zeta(k), \tag{7·6}$$

provided $k > 1$; and generally, for $k > r$,

$$F_k^{(r)}(0) = (1-2^{r-k})\,\Gamma(k+1)\,\zeta(k+1-r). \tag{7·7}$$

Our numerical results show that this relation, (7·7), holds over a wider range of k values than that for which the above discussion is applicable. Thus, it holds for $k = \frac{1}{2}$ with $r = 1, 2, 3$, as may be shown by comparing the values of $F'(0)$, $F''(0)$, $F'''(0)$ given in the table, and obtained by successive differentiation of the $F(\eta)$ values, with those obtained from (7·7), using the zeta function values given by Gram. These values are shown in Table 7·1, which includes $F_k^{(r)}(0)$ for $r = -1, 0, 1, \ldots 5$ calculated from (7·7), the independently calculated value of $\frac{2}{3}F_{\frac{3}{2}}(0)$ (for which the discussion given above is applicable), and also the independently calculated values of $F'(0)$, $F''(0)$ and $F'''(0)$.

This result suggested that an attempt should be made to establish the relation (7·7) for a wider range of k values, and this is done in an Appendix, where an analytic continuation of $F_k(\eta)$ is defined for all $k \neq 0, \pm 1, \pm 2$, etc. By this means, the relation (7·7) is shown to be valid for the sequence of half-odd-integral values of k, in particular, without restriction on r. This makes possible the calculation of $F_k^{(r)}(0)$ values in terms of zeta function values (over the range for which these have been computed), and also gives information as to the behaviour of the functions $F_k^{(r)}(0)$ in the region $\eta \doteqdot 0$ from the known properties of the zeta functions. Thus successive pairs of derivatives, beginning with $F''(0)$ and $F'''(0)$, are alternately positive and negative, and the gradients of the $F_k^{(r)}(\eta)$ curves at $\eta = 0$ are alternately positive and negative in pairs beginning with $F'(\eta)$ and $F''(\eta)$. This information may now be combined with that as to the behaviour of the functions for $\eta \to -\infty$ and $\eta \to +\infty$, to determine the general character of the variation of $F_k^{(r)}(\eta)$ for the whole η range. For example, for the derived functions $F_{\frac{1}{2}}^{(r)}(\eta)$ a maximum first appears in the F'' curve, and, as mentioned above, the higher derivatives have $(r-2)$ zeros. With increasing r, the zeros move to smaller values of η, a zero first appearing for negative η when $r = 4$. For $r \geqslant 2$ and even, there are $(r-2)/2$ zeros in the range $\eta > 0$ and the same number in the range $\eta < 0$; for r odd, there are $(r-1)/2$ zeros for $\eta > 0$, and $(r-3)/2$ for $\eta < 0$.

Table 7·1—Values of $F_{\frac{1}{2}}^{(r)}(0)$ calculated from zeta functions; and, for comparison, the values of $\frac{2}{3}F_{\frac{3}{2}}(0)$, $F'(0)$, $F''(0)$ and $F'''(0)$, calculated independently

r	$\zeta(\frac{3}{2}-r)$	$F_{\frac{1}{2}}^{(r)}(0)$	Values calculated independently
-1	1·341 487 26	0·768 535 89	$\frac{2}{3}F_{\frac{3}{2}}(0) = 0{\cdot}768\ 536$
0	2·612 375 35	0·678 093 90	
1	$-1{\cdot}460\ 354\ 41$	0·536 077 46	$F'(0) = 0{\cdot}536\ 08$
2	$-0{\cdot}207\ 886\ 22$	0·336 859 12	$F''(0) = 0{\cdot}336\ 8^{\cdot}$
3	$-0{\cdot}025\ 485\ 20$	0·105 178 18	$F'''(0) = 0{\cdot}105$
4	0·008 516 93	$-0{\cdot}077\ 847\ 17$	
5	0·004 441 01	$-0{\cdot}085\ 119\ 97$	

Finally, it may be noted that, for small values of $|\eta|$, it is convenient to evaluate $F(\eta)$ and its derivatives from the $F_{\frac{1}{2}}^{(r)}(0)$ values of Table 7·1 by means of a Taylor series expansion; the precision attainable is comparable with that of the listed values for $|\eta| \leqslant 0{\cdot}3$.

8—Description and use of table

The table gives the values of five Fermi-Dirac functions which may be required in applications, namely, $\frac{2}{3}F_{\frac{3}{2}}(\eta)$, $F_{\frac{1}{2}}(\eta) = F$, wF', w^2F'' and w^3F''', for values of η from $-4{\cdot}0$ to $+20{\cdot}0$ at intervals of $w = 0{\cdot}1$ in the argument. The magnitudes are set out to six decimal places for $-4{\cdot}0 \leqslant \eta \leqslant +4{\cdot}0$, and are believed to be correct to within $0^{\cdot}$ and 1 in the sixth place for negative and positive η values, respectively, in this range. For $4{\cdot}0 < \eta \leqslant 20{\cdot}0$, only five decimal places are given, and the values are correct to $0^{\cdot}$ in the fifth place. (The w^3F''' values are not given for $\eta > 8{\cdot}0$ as they are small $(< 1^{\cdot})$, but they may be obtained, if required, from inspection of the w^2F'' values.) The second and subsequent columns of the table give the values of the derivatives (directly, or with the factor w) of the function listed in the preceding column.

The utilization of successive derivatives for interpolation, to which reference has been made, will now be considered more fully. In applications, values of the functions will be required for values of η lying between those for which the functions are listed, and interpolation will be necessary. This may be effected by drawing up a table of differences and using the Bessel formula (2·11); but, more conveniently, a Taylor series expansion gives the value of $f_n = f(a+nw)$ in terms of the listed values of $f_0 = f(a)$ and its derivatives:

$$f_n = f_0 + \sum_{r=1}^{\infty} \frac{n^r}{r!} w^r f_0^{(r)}. \tag{8·1}$$

The tables, therefore, provide all the information required in interpolation of any of the listed functions, and no differencing of the functions is necessary. An estimate of the number of terms of the expansion (8·1) which are required in interpolation is readily made. For $F(\eta)$, for example, putting $n = 1$ shows that by using only three terms in (8·1) the error does not exceed 0˙ in the sixth decimal place throughout the range of η values covered by the table. Since the expansion also applies for negative n, however, the greatest value of $|n|$ necessary is 0·5, and an error of less than 0˙ in the fifth place (which will usually be adequate) is then obtained by using only two terms in (8·1).

The process of inverse interpolation may also be required in applications, and an appropriate formula for determining n when f_n is given may be found by reversing the expansion (8·1). This method is discussed by Becker and van Orstrand (1909, p. xxxix) who give a formula, equivalent, as far as terms in n_1^4, to

$$n = n_1 - \tfrac{1}{2}q_2 n_1^2 + \{\tfrac{1}{2}q_2^2 - \tfrac{1}{6}q_3\} n_1^3 + \{\tfrac{5}{12}q_2 q_3 - \tfrac{5}{8}q_2^3 - \tfrac{1}{24}q_4\} n_1^4, \tag{8·2}$$

where $$n_1 = (f_n - f_0)/wf_0', \quad \text{and} \quad q_r = w^r f_0^{(r)}/wf_0'.$$

This formula applies for both positive and negative values of n_1, so that in practice it is possible to choose the listed value differing least from f_n, and $|n_1|$ need not exceed 0·5.

Although the number of terms of the series (8·2) required to give the value of n as accurately as possible depends upon the diminution in q_r as r increases, the factor primarily determining the precision attainable is actually the number of significant figures in n_1, or equivalently in wf_0'. For $F(\eta)$, for example, the values of q_r are greatest in the neighbourhood of $\eta = -4·0$ (the ratio of successive q_r values tends to 0·1, the tabular interval, as η decreases), but no useful purpose is served by including terms beyond that involving n_1^3 in the determination of a value of n in this region, since the accuracy possible is limited essentially by the precision of the wF' value, 1602˙, namely 0˙. In the neighbourhood of $\eta = -4·0$, therefore, the tables are adequate for the determination of n to within 0·000 1˙ (since $|n_1|$ need not exceed 0·5) so that, given $F(\eta)$,

η may be found to within 0·000 01˙. Similar considerations show that in these inverse calculations the value of η in the range $-4·0$ to $+4·0$ may be determined to within 16, 6, 3, 1 and 0˙ in the sixth place for η greater than -4, -3, -2, -1 and $+1$ respectively, and that only for $-1·0<\eta<+1·0$ is it necessary to include the term involving n_1^4. In the range $4·0<\eta\leqslant 20·0$, where $F(\eta)$ is given to within 0˙ in the fifth decimal place, η may be determined to within 1˙ in the sixth decimal place, and this precision is attained for $\eta\geqslant 4·5$ even if the n_1^3 term is neglected. If the value of η corresponding to a given $F(\eta)$ is required to a lower degree of precision than that to which the above discussion applies, simple approximate methods of inverse interpolation may be devised without difficulty. The result of any inverse interpolation calculation may be checked by means of the expansion (8·1) or by evaluating differences and using the direct interpolation formula of Bessel (2·11).

The formula (8·2) has been applied in producing Table 8·1, which gives the values of η corresponding to a number of values of (kT/ϵ_0), when the relation (1·11) holds; an indication is thus given of the ranges of temperature corresponding to different parts of the $F(\eta)$ table. (Table 8·1 would be useful in dealing numerically with such problems as electronic specific heat and spin paramagnetism to which reference has been made in §1.)

Table 8·1—Corresponding values of (kT/ϵ_0) and η

For the limiting values of η in the $F(\eta)$ table, namely, $\eta=+20·0$ and $-4·0$, (kT/ϵ_0) takes the values 0·049 897 and 11·955 19˙ respectively

kT/ϵ_0	η	kT/ϵ_0	η
0·05	19·958 721˙	1·0	−0·021 461
0·1	9·916 412˙	1·1	−0·199 181
0·2	4·822 880	1·2	−0·357 435
0·3	3·048 607˙	1·3	−0·500 051
0·4	2·100 868	1·4	−0·629 842˙
0·5	1·486 224	1·5	−0·748 929˙
0·6	1·041 445	1·6	−0·858 951˙
0·7	0·696 587	1·7	−0·961 199˙
0·8	0·416 421˙	1·8	−1·056 707˙
0·9	0·181 112	1·9	−1·146 316˙
1·0	−0·021 461	2·0	−1·230 719

Some of the inverse interpolation calculations have been checked independently by using a direct interpolation formula and a trial and error procedure, but little time is saved by adopting this method. If a large number of accurate inverse interpolations had to be carried out, the desirability of adopting the two-machine method developed by Comrie (1936) would be worthy of consideration. In view of the fact that these alternative methods involve the production of differences of the function to be interpolated, we emphasize the point that the table given provides all the information required to permit the interpolation of any of the listed functions by using the reversed series (8·2). We have exemplified the procedure by consideration of inverse interpolation from $F(\eta)$, not only because $F(\eta)$ is the basic computed function, but also because it is almost exclusively for this function that inverse interpolation will be required in physical applications. If necessary, the accuracy attainable in inverse interpolation from the other functions can be determined by an extension of the arguments given above.

The consideration of physical applications which may be made of the computed functions falls outside the scope of this paper; the examples mentioned in § 1 are sufficient to indicate the range of utility. The body of the paper deals essentially with the computation of a series of functions without regard to their application, and the results are presented in a form which seems most appropriate for showing the numerical characteristics of these functions. Since, however, physical considerations provided the incentive for starting this work, it is pertinent to consider whether the form in which the results are given is the most useful for the purpose of applications. In the examples mentioned the values of $F_{\frac{3}{2}}(\eta)/F(\eta)$ and of $F'(\eta)/F(\eta)$ would be required for a series of values of (kT/ϵ_0); it would seem, therefore, that for this purpose, a table giving η, $F_{\frac{3}{2}}$, F, F', ... at equal intervals of (kT/ϵ_0) would be most useful. Such a table could be constructed (at the expense of considerable labour in inverse and direct interpolation), but as an alternative to the table given it would have a number of disadvantages. If it were to contain an equivalent amount of numerical data relevant to interpolation, the necessity of tabulating differences of the various functions would make it very extensive. Moreover, the direct application of the table would be restricted to a relatively narrow range of problems, in which the effect of external fields (using this term in a generalized sense) is zero or small (as in the problems of electronic specific heat or spin paramagnetism); in dealing with more complicated problems (ferromagnetism may be mentioned as an example) the original table would be more suitable. Finally, the occasional convenience of a modified form of presentation of the numerical results is more than offset by the clear manner in which the properties of the basic functions are exhibited in the table as given.

Although there are strong reasons for tabulating the basic Fermi-Dirac functions in a form similar to that adopted rather than in a form appropriate for specialized applications, such applications would often be facilitated by supplementary tables, of which Table 8·1 is an example in skeleton form. The value of an elaborate table of this type would, however, scarcely be commensurate with the labour of its production; for usually in applications only a fairly small number of entries will be required, and these can be calculated, at small expenditure of time, from the basic table. This table not only shows the numerical characteristics of a series of important functions, but it also removes the major part of the computational difficulties involved in the application of Fermi-Dirac statistics to a wide range of problems.

Appendix*

Analytic continuation of the function $F_k(\eta)$

Mention has been made earlier in this paper of the desirability of obtaining an analytic continuation of the function

$$F_k(\eta) = \int_0^\infty \frac{t^k\,dt}{e^{t-\eta}+1}, \tag{A 1}$$

which is adequately defined only for $k > -1$, particularly with a view to establishing the relation

$$F_k^{(r)}(0) = (1-2^{r-k})\,\Gamma(k+1)\,\zeta(k+1-r) \tag{7·7}$$

for a wider range of r values than the criterion $k > r$ allows. It is convenient to modify the notation, and to set

$$G(x, \eta) = \int_0^\infty \frac{t^{x-1}\,dt}{e^{t-\eta}+1}, \tag{A 2}$$

so defining an analytic function when x is positive; then

$$G(x, \eta) = F_k(\eta), \quad \text{when} \quad k = x-1, \tag{A 3}$$

and the integral $F_{\frac{1}{2}}(\eta) = G(\frac{3}{2}, \eta)$, a member of the sequence $G(n+\frac{1}{2}, \eta)$. This notation is more appropriate than that usually employed (and for that reason used in this paper), as is illustrated by the limiting forms: for $\eta \gg 1$, $G(x, \eta) \to x^{-1}\eta^x$; for $-\eta \gg 1$, $G(x, \eta) \to e^\eta \Gamma(x)$.

A continuation of the function (A 2) appropriate to negative values of x is provided (except at certain points) by considering the function

$$G(z, \eta) = \int_0^\infty \frac{t^{z-1}\,dt}{e^{t-\eta}+1}, \tag{A 4}$$

which is identical with (A 2) for z real and positive. By a procedure similar to that for obtaining Hankel's expression for $\Gamma(z)$ (Whittaker and Watson 1935, para 12·22, p. 244), it may be shown that when the real part of z is positive and not integral

$$G(z, \eta) = -\frac{1}{2i\sin\pi z}\int_C \frac{(-t)^{z-1}\,dt}{e^{t-\eta}+1}, \tag{A 5}$$

where the path of integration, C, starts at infinity on the real axis, encircles the origin in a positive direction, and returns to infinity, and is so chosen that it does not include any of the points $\pm(2n+1)\pi i+\eta$. Adopting the symbolism of Whittaker and Watson for this contour, (A 5) may be written

$$G(z, \eta) = -\frac{1}{2i\sin\pi z}\int_\infty^{(0+)} \frac{(-t)^{z-1}\,dt}{e^{t-\eta}+1}, \tag{A 6}$$

* We are indebted to Mr C. W. Gilham for verifying the general treatment given in this Appendix.

which defines an analytic function for all values of z except $0, \pm 1, \pm 2, \ldots$, and, except for these values, is an appropriate extension of $G(x, \eta)$. A special treatment may be developed for the integral values of z, but this need not be considered here.

Relation between $G(z, \eta)$ and $G(z-1, \eta)$

It may now be shown that the relation

$$F_k'(\eta) = kF_{k-1}(\eta), \tag{6·3}$$

or

$$\frac{\partial}{\partial \eta} G(x, \eta) = (x-1)\, G(x-1, \eta),$$

whose range of application is restricted to $k>0$, or $x>1$, in virtue of the definitions (A 1) and (A 2), holds in the form

$$\frac{\partial}{\partial \eta} G(z, \eta) = (z-1)\, G(z-1, \eta), \tag{A 7}$$

for all values of z which are not real and integral. For

$$2i \sin \pi(z-1)\, G(z-1, \eta) = -\int_\infty^{(0+)} \frac{(-t)^{z-2}\, dt}{e^{t-\eta}+1}$$

$$= \left\{\frac{2it^{z-1} \sin \pi(z-1)}{(z-1)\,(e^{t-\eta}+1)}\right\}_{t\to\infty} + \int_\infty^{(0+)} \frac{1}{z-1} \cdot \frac{(-t)^{z-1} e^{t-\eta}\, dt}{(e^{t-\eta}+1)^2}$$

$$= -\frac{1}{z-1}\, 2i \sin \pi z \frac{\partial}{\partial \eta} G(z, \eta).$$

Thus

$$\frac{\partial}{\partial \eta} G(z, \eta) = -\frac{2i \sin \pi(z-1)}{2i \sin \pi z} (z-1)\, G(z-1, \eta),$$

and (A 7) follows. Therefore the relation (6·3), which has been applied in this paper to the sequence $k = n+\frac{1}{2}$, where n is integral, is not restricted to positive values of n, provided $F_k(\eta)$ is suitably defined.

Relation between $G(z, 0)$ and the Riemann zeta function

The Riemann zeta function may be expressed (Whittaker and Watson 1935, p. 266) in the form

$$\zeta(z) = -\frac{\Gamma(1-z)}{2\pi i} \int_\infty^{(0+)} \frac{(-t)^{z-1}\, dt}{e^t-1}. \tag{A 8}$$

Hence, using the identity

$$\frac{1}{e^t+1} = \frac{1}{e^t-1} - \frac{2}{e^{2t}-1}, \tag{7·4}$$

it follows that
$$G(z,\,0) = -\frac{1}{2i\sin\pi z}\int_{\infty}^{(0+)}\frac{(-t)^{z-1}\,dt}{e^t+1}$$
$$= -\frac{1}{2i\sin\pi z}\,(1-2^{1-z})\int_{\infty}^{(0+)}\frac{(-t)^{z-1}\,dt}{e^t-1}$$
$$= \frac{\pi}{\sin\pi z}\,(1-2^{1-z})\,\frac{\zeta(z)}{\Gamma(1-z)},$$

which, by the use of the functional relation $\Gamma(z)\,\Gamma(1-z) = \pi/\sin\pi z$, becomes
$$G(z,\,0) = \Gamma(z)\,(1-2^{1-z})\,\zeta(z). \qquad \text{(A 9)}$$

Repeated application of (A 7) and subsequent use of (A 9) gives
$$\left(\frac{\partial}{\partial\eta}\right)^r G(z,\,0) = (z-1)\,(z-2)\,\ldots\,(z-r)\,G(z-r,\,0)$$
$$= \frac{\Gamma(z)}{\Gamma(z-r)}\,G(z-r,\,0),$$
i.e.
$$\left(\frac{\partial}{\partial\eta}\right)^r G(z,\,0) = \Gamma(z)\,(1-2^{1-z+r})\,\zeta(z-r), \qquad \text{(A 10)}$$

showing that by suitably defining $F_k(\eta)$, the relation (7·7) holds for both positive and negative half-odd-integral values of k, in particular, without restriction on r, the order of differentiation.

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TABLE OF FERMI-DIRAC FUNCTIONS

The table gives the values of the functions

$$\tfrac{2}{3}F_{\frac{3}{2}}(\eta),\ F_{\frac{1}{2}}(\eta) = F(\eta) = F,\ wF',\ w^2F'' \text{ and } w^3F''',$$

where $F_k(\eta) = \int_0^\infty \frac{x^k\,dx}{e^{x-\eta}+1}$, and $F' = \frac{d}{d\eta}F(\eta)$, at intervals $w = 0{\cdot}1$ in the argument. Since $\frac{d}{d\eta}\{\tfrac{2}{3}F_{\frac{3}{2}}(\eta)\} = F(\eta)$, each column after the first gives a multiple of a derivative of each of the functions given in the preceding columns.

The functions are listed to the sixth decimal place for $-4{\cdot}0 \leqslant \eta \leqslant +4{\cdot}0$, and to the fifth decimal place for $4{\cdot}0 \leqslant \eta \leqslant 20{\cdot}0$. The dot symbol $^{\bullet}$ after the last printed digit indicates that the next digit lies between 3 and 7.

The values are believed to be correct to within $0^{\bullet}$ and 1 in the sixth decimal place in the ranges $-4{\cdot}0 \leqslant \eta \leqslant 0{\cdot}0$ and $0{\cdot}0 < \eta \leqslant 4{\cdot}0$, respectively, and to within $0^{\bullet}$ in the fifth place for $4{\cdot}0 < \eta \leqslant 20{\cdot}0$.

Suitable methods for direct and inverse interpolation are described in § 8.

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η	$\frac{2}{3}F_{\frac{3}{2}}$	F	wF'	w^2F''	w^3F'''
−4·0	0·016 179˙	0·016 128	1 602˙	158	15˙
−3·9	0·017 875	0·017 812	1 768˙	174˙	17
−3·8	0·019 748	0·019 670˙	1 952	192	18˙
−3·7	0·021 816	0·021 721˙	2 153˙	211˙	20˙
−3·6	0·024 099	0·023 984˙	2 376	233	22˙
−3·5	0·026 620˙	0·026 480˙	2 620˙	256˙	24˙
−3·4	0·029 404	0·029 233˙	2 889˙	282˙	27
−3·3	0·032 476˙	0·032 269	3 186	310˙	29˙
−3·2	0·035 868	0·035 615	3 511˙	341˙	32˙
−3·1	0·039 611	0·039 303	3 870	375	35˙
−3·0	0·043 741	0·043 366˙	4 263	412	38˙
−2·9	0·048 298	0·047 842	4 695	452	42
−2·8	0·053 324˙	0·052 770	5 168˙	496	45˙
−2·7	0·058 868˙	0·058 194	5 687˙	543	49˙
−2·6	0·064 981˙	0·064 161˙	6 256	595	54
−2·5	0·071 720˙	0·070 724˙	6 879	651	58
−2·4	0·079 148	0·077 938˙	7 559˙	711	63
−2·3	0·087 332	0·085 864	8 303	776˙	68
−2·2	0·096 347	0·094 566˙	9 114	846˙	73
−2·1	0·106 273˙	0·104 116	9 997˙	922	78
−2·0	0·117 200˙	0·114 588	10 959˙	1 003	83˙
−1·9	0·129 224˙	0·126 063	12 005	1 089	89
−1·8	0·142 449˙	0·138 627˙	13 139	1 180˙	94˙
−1·7	0·156 989˙	0·152 373	14 368	1 278	100
−1·6	0·172 967	0·167 397	15 697	1 381	105˙
−1·5	0·190 515	0·183 802	17 131	1 489	110˙
−1·4	0·209 777	0·201 696	18 676	1 602	116
−1·3	0·230 907˙	0·221 193	20 337	1 720˙	120˙
−1·2	0·254 073	0·242 410˙	22 118	1 843	124˙
−1·1	0·279 451	0·265 471	24 024	1 969	128
−1·0	0·307 232˙	0·290 501	26 057˙	2 098˙	131
−0·9	0·337 621	0·317 630	28 222	2 231	133
−0·8	0·370 833	0·346 989˙	30 520	2 364˙	134
−0·7	0·407 098	0·378 714	32 951˙	2 499	134˙
−0·6	0·446 659	0·412 937	35 517˙	2 633	133˙
−0·5	0·489 773	0·449 793	38 217	2 766	131˙
−0·4	0·536 710	0·489 414˙	41 048	2 896	128˙
−0·3	0·587 752˙	0·531 931˙	44 007˙	3 022	124
−0·2	0·643 197	0·577 470˙	47 091	3 144	119
−0·1	0·703 351	0·626 152˙	50 293	3 259˙	112˙
0·0	0·768 536	0·678 094	53 608	3 368˙	105

η	$\frac{2}{3}F_{\frac{3}{2}}$	F	wF'	w^2F''	w^3F'''
0·0	0·768 536	0·678 094	53 608	3 368˙	105
0·1	0·839 082	0·733 403	57 027˙	3 470	97
0·2	0·915 332	0·792 181˙	60 544	3 562	88
0·3	0·997 637	0·854 521	64 149	3 646	78˙
0·4	1·086 358	0·920 505˙	67 832˙	3 719˙	69
0·5	1·181 862˙	0·990 209	71 584˙	3 783	58˙
0·6	1·284 526	1·063 694˙	75 395	3 836˙	48
0·7	1·394 729	1·141 015˙	79 254	3 880	38
0·8	1·512 858	1·222 215˙	83 151˙	3 913	28
0·9	1·639 302˙	1·307 327˙	87 076˙	3 936	18˙
1·0	1·774 455	1·396 375	91 020˙	3 950	9˙
1·1	1·918 709˙	1·489 372	94 974	3 955	1
1·2	2·072 461	1·586 323˙	98 928	3 952	−7
1·3	2·236 106	1·687 226	102 875	3 941	−14˙
1·4	2·410 037˙	1·792 068˙	106 807˙	3 923	−21˙
1·5	2·594 650	1·900 833˙	110 718˙	3 898˙	−27˙
1·6	2·790 334˙	2·013 496˙	114 602	3 868˙	−32˙
1·7	2·997 478˙	2·130 027	118 453˙	3 833˙	−37
1·8	3·216 467˙	2·250 391	122 267˙	3 794˙	−41
1·9	3·447 683	2·374 548˙	126 041	3 751˙	−44˙
2·0	3·691 502	2·502 458	129 770	3 706	−47
2·1	3·948 298	2·634 072˙	133 451˙	3 657˙	−49
2·2	4·218 438˙	2·769 344˙	137 084	3 607˙	−50˙
2·3	4·502 287	2·908 224	140 666	3 556	−52
2·4	4·800 202	3·050 659˙	144 196	3 504	−53
2·5	5·112 536	3·196 598˙	147 673	3 450˙	−53˙
2·6	5·439 637	3·345 988	151 097	3 397	−53˙
2·7	5·781 847	3·498 775	154 467˙	3 343˙	−53˙
2·8	6·139 503	3·654 905˙	157 784˙	3 290˙	−53
2·9	6·512 937˙	3·814 326˙	161 049	3 238	−52
3·0	6·902 476˙	3·976 985˙	164 261	3 186	−51˙
3·1	7·308 441	4·142 831	167 421	3 135	−50˙
3·2	7·731 147	4·311 811	170 531	3 085	−50
3·3	8·170 906	4·483 876˙	173 591	3 035˙	−48˙
3·4	8·628 023˙	4·658 977˙	176 602˙	2 987˙	−47˙
3·5	9·102 801	4·837 066	179 566˙	2 941	−46
3·6	9·595 535	5·018 095	182 484˙	2 895	−45
3·7	10·106 516˙	5·202 020	185 357	2 850˙	−44
3·8	10·636 034	5·388 795	188 186	2 807˙	−42˙
3·9	11·184 369	5·578 378	190 972˙	2 765˙	−41
4·0	11·751 801˙	5·770 726˙	193 717˙	2 725	−40

η	$\frac{2}{3}F_{\frac{3}{2}}$	F	wF'	w^2F''	w^3F''
4·0	11·751 80	5·770 72˙	193 72	2 72˙	−4
4·1	12·338 60˙	5·965 80	196 42	2 68˙	−4
4·2	12·945 05	6·163 56	199 09	2 65	−4
4·3	13·571 40˙	6·363 96˙	201 72	2 61	−3˙
4·4	14·217 93	6·566 98	204 31	2 57˙	−3˙
4·5	14·884 89	6·772 57˙	206 87	2 54	−3˙
4·6	15·572 53	6·980 70˙	209 39	2 50˙	−3˙
4·7	16·281 11	7·191 34˙	211 88	2 47˙	−3
4·8	17·010 88	7·404 45˙	214 34	2 44	−3
4·9	17·762 08˙	7·620 01	216 76˙	2 41	−3
5·0	18·534 96˙	7·837 97˙	219 16˙	2 38	−3
5·1	19·329 76	8·058 32˙	221 53	2 35˙	−3
5·2	20·146 71	8·281 03	223 87	2 32˙	−3
5·3	20·986 04	8·506 06	226 18˙	2 30	−2˙
5·4	21·847 99˙	8·733 39	228 47	2 27˙	−2˙
5·5	22·732 79˙	8·962 99˙	230 73˙	2 25	−2˙
5·6	23·640 67	9·194 85	232 97	2 22˙	−2˙
5·7	24·571 84	9·428 93	235 18˙	2 20	−2˙
5·8	25·526 53	9·665 21	237 37˙	2 18	−2
5·9	26·504 95˙	9·903 67	239 54	2 16	−2
6·0	27·507 33˙	10·144 28˙	241 69	2 13˙	−2
6·1	28·533 88˙	10·387 03˙	243 81	2 11˙	−2
6·2	29·584 81˙	10·631 90	245 91˙	2 09˙	−2
6·3	30·660 33˙	10·878 86	248 00	2 07˙	−2
6·4	31·760 65˙	11·127 89˙	250 06˙	2 05˙	−2
6·5	32·885 98	11·378 98˙	252 11	2 03˙	−2
6·6	34·036 52	11·632 11˙	254 14	2 02	−2
6·7	35·212 47	11·887 26	256 15	2 00	−2
6·8	36·414 04	12·144 40˙	258 14	1 98˙	−2
6·9	37·641 42	12·403 54	260 12	1 97	−1˙
7·0	38·894 81	12·664 64	262 08	1 95	−1˙
7·1	40·174 41	12·927 69	264 02	1 93˙	−1˙
7·2	41·480 41˙	13·192 67˙	265 95	1 92	−1˙
7·3	42·813 01	13·459 58	267 86	1 90˙	−1˙
7·4	44·172 39˙	13·728 39˙	269 76	1 89	−1˙
7·5	45·558 75	13·999 10	271 64˙	1 87˙	−1˙
7·6	46·972 27˙	14·271 68	273 51˙	1 86	−1˙
7·7	48·413 15	14·546 12	275 37	1 85	−1˙
7·8	49·881 56	14·822 41	277 21	1 83˙	−1˙
7·9	51·377 69˙	15·100 53˙	279 04	1 82	−1
8·0	52·901 73	15·380 48˙	280 85˙	1 81	−1

η	$\frac{2}{3}F_{\frac{3}{2}}$	F	wF'	w^2F''
8·0	52·901 73	15·380 48˙	280 85˙	1 81
8·1	54·453 85	15·662 24˙	282 66	1 80
8·2	56·034 24	15·945 80	284 45	1 78˙
8·3	57·643 07	16·231 14˙	286 23	1 77˙
8·4	59·280 52˙	16·518 26	288 00	1 76
8·5	60·946 78	16·807 14	289 75˙	1 75
8·6	62·642 01	17·097 76˙	291 50	1 74
8·7	64·366 39	17·390 13	293 23˙	1 73
8·8	66·120 09˙	17·684 23	294 95˙	1 72
8·9	67·903 29˙	17·980 04	296 67	1 71
9·0	69·716 16	18·277 56	298 37	1 69˙
9·1	71·558 86˙	18·576 77˙	300 06	1 68˙
9·2	73·431 57	18·877 68	301 74˙	1 67˙
9·3	75·334 45˙	19·180 26	303 41˙	1 67
9·4	77·267 68	19·484 51	305 08	1 66
9·5	79·231 41	19·790 41	306 73	1 65
9·6	81·225 82	20·097 96˙	308 37	1 64
9·7	83·251 06	20·407 15˙	310 01	1 63
9·8	85·307 30	20·717 97˙	311 63	1 62
9·9	87·394 71	21·030 42	313 25	1 61
10·0	89·513 44	21·344 47	314 86	1 60˙
10·1	91·663 65˙	21·660 13	316 45˙	1 59˙
10·2	93·845 52	21·977 38	318 04˙	1 58˙
10·3	96·059 18˙	22·296 22	319 63	1 58
10·4	98·304 81˙	22·616 64	321 20˙	1 57
10·5	100·582 56˙	22·938 62˙	322 77	1 56
10·6	102·892 59	23·262 17˙	324 33	1 55˙
10·7	105·235 05	23·587 28	325 88	1 54˙
10·8	107·610 10	23·913 93	327 42˙	1 54
10·9	110·017 89	24·242 12˙	328 96	1 53
11·0	112·458 57˙	24·571 84˙	330 48˙	1 52˙
11·1	114·932 31	24·903 09˙	332 01	1 52
11·2	117·439 24˙	25·235 86	333 52	1 51
11·3	119·979 53	25·570 13˙	335 03	1 50˙
11·4	122·553 32	25·905 91˙	336 53	1 49˙
11·5	125·160 76˙	26·243 19	338 02	1 49
11·6	127·802 01	26·581 95˙	339 51	1 48
11·7	130·477 20˙	26·922 20˙	340 98˙	1 47˙
11·8	133·186 50	27·263 93	342 46	1 47
11·9	135·930 04	27·607 12	343 92˙	1 46˙
12·0	138·707 97˙	27·951 78	345 38˙	1 45˙

η	$\frac{2}{3}F_{\frac{3}{2}}$	F	wF'	w^2F''
12·0	138·707 97˙	27·951 78	345 38˙	1 45˙
12·1	141·520 44˙	28·297 89	346 84	1 45
12·2	144·367 60	28·645 45˙	348 29	1 44˙
12·3	147·249 58˙	28·994 46˙	349 73	1 44
12·4	150·166 54	29·344 91	351 16˙	1 43
12·5	153·118 61˙	29·696 79	352 59˙	1 42˙
12·6	156·105 94˙	30·050 09˙	354 01˙	1 42
12·7	159·128 68	30·404 82	355 43˙	1 41˙
12·8	162·186 96.	30·760 96	356 84˙	1 41
12·9	165·280 92	31·118 51	358 25	1 40˙
13·0	168·410 71	31·477 46˙	359 65	1 40
13·1	171·576 46	31·837 81˙	361 05	1 39
13·2	174·778 31˙	32·199 56	362 43˙	1 38˙
13·3	178·016 42	32·562 68˙	363 82	1 38
13·4	181·290 90	32·927 20	365 20	1 37˙
13·5	184·601 90	33·293 08˙	366 57	1 37
13·6	187·949 56	33·660 34	367 94	1 36˙
13·7	191·334 01˙	34·028 96	369 30˙	1 36
13·8	194·755 40	34·398 94˙	370 66	1 35˙
13·9	198·213 85	34·770 28	372 01˙	1 35
14·0	201·709 50	35·142 97	373 36	1 34˙
14·1	205·242 49	35·517 00˙	374 70˙	1 34
14·2	208·812 95	35·892 38	376 04	1 33˙
14·3	212·421 01	36·269 08˙	377 37˙	1 33
14·4	216·066 81	36·647 12˙	378 70˙	1 32˙
14·5	219·750 48	37·026 49	380 03	1 32
14·6	223·472 15	37·407 18	381 34˙	1 31˙
14·7	227·231 96	37·789 18	382 66	1 31
14·8	231·030 03	38·172 50	383 97	1 31
14·9	234·866 50	38·557 12	385 27˙	1 30˙
15·0	238·741 50	38·943 04˙	386 57˙	1 30
15·1	242·655 15˙	39·330 27	387 87	1 29˙
15·2	246·607 59˙	39·718 79	389 16˙	1 29
15·3	250·598 95˙	40·108 59˙	390 45	1 28˙
15·4	254·629 36	40·499 69	391 73˙	1 28
15·5	258·698 93˙	40·892 06˙	393 01˙	1 27˙
15·6	262·807 81˙	41·285 71˙	394 29	1 27
15·7	266·956 12	41·680 64	395 56	1 27
15·8	271·143 98˙	42·076 83	396 82˙	1 26˙
15·9	275·371 53	42·474 29	398 09	1 26
16·0	279·638 88˙	42·873 00˙	399 34˙	1 25˙

η	$\frac{2}{3}F_{\frac{3}{2}}$	F	wF'	w^2F''
16·0	279·638 88˙	42·873 00˙	399 34˙	1 25˙
16·1	283·946 17	43·272 98	400 60	1 25
16·2	288·293 52	43·674 20˙	401 85	1 25
16·3	292·681 05˙	44·076 68	403 09˙	1 24˙
16·4	297·108 90	44·480 39˙	404 34	1 24
16·5	301·577 17˙	44·885 35˙	405 58	1 23˙
16·6	306·086 01	45·291 55	406 81	1 23˙
16·7	310·635 53	45·698 98	408 04˙	1 23
16·8	315·225 85	46·107 63˙	409 27	1 22˙
16·9	319·857 09˙	46·517 52	410 49˙	1 22
17·0	324·529 39	46·928 62˙	411 71˙	1 22
17·1	329·242 86	47·340 95	412 93	1 21˙
17·2	333·997 62	47·754 48˙	414 14˙	1 21
17·3	338·793 80	48·169 23˙	415 35	1 21
17·4	343·631 51	48·585 19	416 56	1 20˙
17·5	348·510 87˙	49·002 35	417 76	1 20
17·6	353·432 02	49·420 71	418 96	1 19˙
17·7	358·395 06	49·840 26˙	420 15˙	1 19˙
17·8	363·400 11	50·261 01	421 34˙	1 19
17·9	368·447 30	50·682 95	422 53˙	1 18˙
18·0	373·536 74	51·106 08	423 72	1 18˙
18·1	378·668 55˙	51·530 39	424 90	1 18
18·2	383·842 86	51·955 87˙	426 08	1 17˙
18·3	389·059 77	52·382 54	427 25	1 17˙
18·4	394·319 40˙	52·810 38	428 42˙	1 17
18·5	399·621 88˙	53·239 39	429 59	1 17
18·6	404·967 32	53·669 56˙	430 76	1 16˙
18·7	410·355 83˙	54·100 90	431 92	1 16
18·8	415·787 54	54·533 40	433 08	1 16
18·9	421·262 55˙	54·967 06	434 23˙	1 15˙
19·0	426·780 99	55·401 87	435 38˙	1 15
19·1	432·342 97	55·837 83	436 53˙	1 15
19·2	437·948 59˙	56·274 94	437 68˙	1 14˙
19·3	443·597 99˙	56·713 20	438 82˙	1 14
19·4	449·291 27˙	57·152 59˙	439 97	1 14
19·5	455·028 55	57·593 13	441 10˙	1 13˙
19·6	460·809 94	58·034 80˙	442 24	1 13˙
19·7	466·635 55	58·477 61	443 37	1 13
19·8	472·505 50	58·921 54˙	444 50	1 13
19·9	478·419 89˙	59·366 61	445 62˙	1 12˙
20·0	484·378 85˙	59·812 79˙	446 75	1 12

Summary

The application of Fermi-Dirac statistics to physical problems (examples of which are indicated) requires the evaluation of integrals of the form $F_k(\eta) = \int_0^\infty \{x^k/(e^{x-\eta}+1)\}\,dx$, especially for $k = \frac{1}{2}$ and $k = \frac{3}{2}$, and of a number of related functions.

This paper is primarily concerned with the evaluation of $F_{\frac{1}{2}}(\eta) = F$, from which the other functions may be obtained, for a wide range of values of the argument η. Series expansions, which are available for $\eta \gg 1$ and $\eta < 0$, corresponding to $\epsilon_0/kT \gg 1$ and approximately $\epsilon_0/kT < 1$ (ϵ_0 being the maximum particle energy in the Fermi-Dirac distribution at absolute zero), are studied in detail and are employed in the calculation of F for $\eta \geqslant 16{\cdot}0$ and $\eta < 0{\cdot}0$. The determination of $F(0)$ is carried out by means of a relation between the functions $F_k(0)$ and the Riemann zeta functions. For values of η between 0 and 16, the computations are made by numerical integration methods, supplemented by the use of series for the initial and final parts of the x range. A direct method is used for $0{\cdot}0 < \eta < 3{\cdot}0$, but for $3{\cdot}0 \leqslant \eta \leqslant 16{\cdot}0$, a modified procedure greatly reduces the work of computation.

From the $F_{\frac{1}{2}}(\eta)$ table so obtained, values of $F_{\frac{3}{2}}(\eta)$ are found by numerical integration, and of the derivatives F', F'' and F''' by numerical differentiation. The final table gives, at tabular intervals $w = 0{\cdot}1$, the values of the functions $\frac{2}{3}F_{\frac{3}{2}}(\eta)$, $F_{\frac{1}{2}}(\eta) = F$, wF', w^2F'' and w^3F''', to six decimal places for $-4{\cdot}0 \leqslant \eta \leqslant +4{\cdot}0$, and to five decimal places for $4{\cdot}0 \leqslant \eta \leqslant 20{\cdot}0$. Convenient methods for direct and inverse interpolation are described.

Some properties of the $F_k(\eta)$ functions, defined only for $(k+1)$ positive, are discussed, and an analytic continuation of the functions, obtained in an Appendix, enables these properties to be established for a wider range of k values.

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